

Verification — Linear Time History of a Single Oscillator

FEM Calculator time-history analysis — modal superposition and Newmark direct integration — checked against the closed-form response of a damped single-degree-of-freedom oscillator

Structure type	Massless steel cantilever with a 2 t mass at its top: one degree of freedom along X, $k = 3EI/L^3$
Verification method	Closed-form SDOF solutions: free vibration, damped step load, harmonic load from rest (transient + steady state), step ground acceleration. Newmark's average-acceleration recurrence written out by hand, and its apparent frequency $\bar{\omega} = (2/\Delta t) \cdot \text{atan}(\omega \Delta t/2)$
Quantities checked	Displacement histories, peaks, absolute acceleration, base shear (with the damper force), the frequency measured from the free decay; both integration methods; time-step convergence
Result	Exact match — modal superposition equals the closed forms to round-off for piecewise-linear loading; Newmark equals its hand recurrence to round-off; both converge to the continuous solution as Δt^2

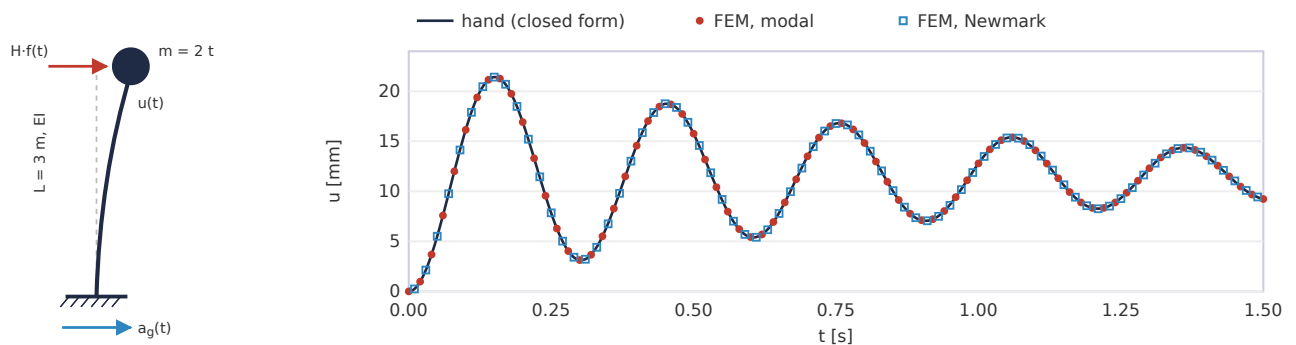


Fig. 1 — Left: the oscillator, a load $H \cdot f(t)$ at the mass or a ground acceleration $a_g(t)$. Right: TH2, $H = 10$ kN applied suddenly with 5 % damping — the closed form (line) and the FEM displacement of the mass with both methods ($\Delta t = 5$ ms).

1. Structure and loading

A single steel column, $L = 3$ m, MSH-H 200×200×8 (EN 10210 hot-finished square hollow section, S235, $E = 210$ GPa, $I = 3\,709.0$ cm⁴), fixed at its base and free at its top, modelled as one member. Its self weight is left out of the mass, so the column is massless and the whole vibrating mass, $m = 2$ t, sits at the top: a load case of $F_z = -19.62$ kN at node 2 is the DYN combination the time-history runs take their mass from ($m = |F_z|/g$, $g = 9.81$ m/s²). A second load case is the horizontal force $H = 10$ kN at the top. The mass acts in X, Y and Z, but only X is excited; with nothing driving Y and Z, the system is a single-degree-of-freedom oscillator in X.

The project file holds four time-history runs, each with $\Delta t = 5$ ms and no background (standing) load:

Run	Excitation	Damping	Duration
TH1	release: starts at rest in the static deflection of $H = 10$ kN, which is removed at $t = 0$	none	3 s
TH2	$H = 10$ kN applied suddenly at $t = 0$ (step time function)	$\zeta = 5 \%$	3 s
TH3	$H \cdot \sin \Omega t$, $f_{\Omega} = 2.64864$ Hz ($\approx 0.8 \cdot f$), from rest	$\zeta = 5 \%$	8 s
TH4	ground acceleration along X, $a_g = 1$ m/s ² from $t = 0$ (a constant record)	none	2 s

2. Hand solution

Data: $L = 3$ m, $E = 210 \cdot 10^6$ kN/m², $I = 3.709015 \cdot 10^{-5}$ m⁴, $m = 2$ t, $H = 10$ kN. Units: kN, m, s, t (1 t = 1 kN·s²/m); displacements in mm.

2.1 The oscillator. The cubic beam element is exact for a cantilever loaded at its end, so the FEM stiffness at the top is the textbook one:

$$EI = 210 \cdot 10^6 \cdot 3.709015 \cdot 10^{-5} = 7788.93 \text{ kN} \cdot \text{m}^2; \quad k = 3EI/L^3 = 3 \cdot 7788.93 / 27 = \mathbf{865.4368} \quad (1)$$

kN/m

$$\omega = \sqrt{(k/m)} = \sqrt{(865.4368 / 2)} = \mathbf{20.801884 \text{ rad/s}}; \quad f = \omega/2\pi = \mathbf{3.310723 \text{ Hz}}; \quad T = 302.0489 \text{ ms} \quad (2)$$

$$\text{static deflection under } H: u_{st} = H/k = 10 / 865.4368 = \mathbf{11.554859 \text{ mm}} \quad (3)$$

2.2 Damping. Modal superposition damps the mode with ζ directly. The Newmark runs use Rayleigh damping $C = a_M \cdot M + b_K \cdot K$ with $f_1 = f_2 = 3.3108$ Hz (the frequency rounded as a user would type it). With $f_1 = f_2$ the coefficients are $a_M = \zeta \omega_1$ and $b_K = \zeta / \omega_1$:

$$\omega_1 = 2\pi \cdot 3.3108 = 20.802370 \text{ rad/s}; \quad a_M = 1.040118 \text{ s}^{-1}; \quad b_K = 0.0024036 \text{ s}$$

$$c = a_M \cdot m + b_K \cdot k = 1.040118 \cdot 2 + 0.0024036 \cdot 865.4368 = \mathbf{4.160377 \text{ kN} \cdot \text{s/m}} \quad (= 2\zeta m \omega \text{ to } 3 \cdot 10^{-10}) \quad (4)$$

The $b_K \cdot K$ part is a damper inside the column, so its force passes through the support: the base shear under Newmark is $R_x = -k \cdot (u + b_K \cdot \dot{u})$, while the $a_M \cdot M$ part acts between the mass and the ground.

2.3 Newmark's method by hand. Average acceleration ($\gamma = 1/2$, $\beta = 1/4$), in the incremental form of Chopra, *Dynamics of Structures*, §5.4:

$$\hat{k} = k + 2c/\Delta t + 4m/\Delta t^2 = 865.4368 + 2 \cdot 4.160377/0.005 + 4 \cdot 2/0.005^2 = \mathbf{322\,529.5876\,kN/m} \quad (5)$$

$$\Delta \hat{p}_i = \Delta p_i + (4m/\Delta t + 2c) \cdot \dot{u}_i + 2m \cdot \ddot{u}_i; \quad \Delta u_i = \Delta \hat{p}_i / \hat{k}; \quad \Delta \dot{u}_i = 2\Delta u_i / \Delta t - 2\dot{u}_i; \quad \Delta \ddot{u}_i = 4(\Delta u_i - \dot{u}_i \Delta t) / \Delta t^2 - 2\ddot{u}_i \quad (6)$$

TH2, start: $u_0 = \dot{u}_0 = 0$, $\ddot{u}_0 = (p_0 - c \cdot \dot{u}_0 - k \cdot u_0)/m = 10/2 = 5\,m/s^2$

step 1: $\Delta \hat{p} = 0 + 0 + 2 \cdot 2 \cdot 5 = 20\,kN \rightarrow u_1 = 20/322\,529.5876 = \mathbf{0.062009815\,mm}$, $\dot{u}_1 = 0.024803926\,m/s$, $\ddot{u}_1 = 4.921570372\,m/s^2$

step 2: $u_2 = \mathbf{0.246733799\,mm}$ ($t = 0.01\,s$) ... and so on to the end of the run

Undamped, the recurrence has a closed form: it is unconditionally stable and keeps the amplitude exactly, but it vibrates at an apparent frequency below ω — the period elongation of the method:

$$\bar{\omega} = (2/\Delta t) \cdot \text{atan}(\omega \Delta t / 2) = 400 \cdot \text{atan}(0.0520047) = \mathbf{20.783162\,rad/s}; \quad \bar{\omega}/\omega - 1 = -9 \cdot 10^{-4} \approx -(\omega \Delta t)^2 / 12 = -9 \cdot 10^{-4} \quad (7)$$

2.4 TH1, release. $m \cdot \ddot{u} + k \cdot u = 0$, $u(0) = u_{st}$, $\dot{u}(0) = 0$:

$$\text{exact: } u(t) = u_{st} \cdot \cos \omega t; \quad \text{Newmark: } u_n = u_{st} \cdot \cos \omega \bar{t}_n; \quad \text{peak acceleration } \omega^2 u_{st} = H/m = 5\,m/s^2 \quad (8)$$

2.5 TH2, step load. $m \cdot \ddot{u} + c \cdot \dot{u} + k \cdot u = H$ for $t \geq 0$, from rest, $\omega_d = \omega \sqrt{1 - \zeta^2} = 20.775866\,rad/s$:

$$u(t) = u_{st} \cdot [1 - e^{-\zeta \omega t} \cdot (\cos \omega_d t + \zeta / \sqrt{1 - \zeta^2} \cdot \sin \omega_d t)] \quad (9)$$

$$\text{peak at } t = \pi / \omega_d = 0.151214\,s: u_{\max} = u_{st} \cdot (1 + e^{-\pi \zeta / \sqrt{1 - \zeta^2}}) = 11.554859 \cdot 1.854468 = \mathbf{21.428115\,mm} \quad (10)$$

2.6 TH3, harmonic load. $m \cdot \ddot{u} + c \cdot \dot{u} + k \cdot u = H \cdot \sin \Omega t$ from rest, $\Omega = 2\pi \cdot 2.64864 = 16.641896\,rad/s$, $r = \Omega / \omega = 0.8000187$. Steady state plus the free vibration that makes the start at rest:

$$U = u_{st} / \sqrt{(1 - r^2)^2 + (2\zeta r)^2} = \mathbf{31.334953\,mm} \quad (\text{amplification } 2.711842); \quad \varphi = \text{atan}(2\zeta r / (1 - r^2)) = 0.2186915\,rad \quad (11)$$

$$u(t) = U \cdot \sin(\Omega t - \varphi) + e^{-\zeta \omega t} \cdot (A \cdot \cos \omega_d t + B \cdot \sin \omega_d t) \quad (12)$$

$$u(0) = 0 \rightarrow A = U \cdot \sin \varphi = 6.798194\,mm; \quad \dot{u}(0) = 0 \rightarrow B = (\zeta \omega A - U \Omega \cdot \cos \varphi) / \omega_d = -24.161781\,mm$$

2.7 TH4, ground acceleration. Relative to the ground, $m \cdot \ddot{u} + k \cdot u = -m \cdot a_g$ with $a_g = 1\,m/s^2$, from rest:

$$u(t) = -(a_g/\omega^2) \cdot (1 - \cos \omega t); \text{ peak } |u| = 2a_g/\omega^2 = 2m \cdot a_g/k = \mathbf{4.621944 \text{ mm}} \quad (13)$$

$$\text{absolute acceleration } \ddot{u} + a_g = -\omega^2 \cdot u \rightarrow \text{peak } \mathbf{2a_g = 2 \text{ m/s}^2}; \text{ base shear } k \cdot |u| \rightarrow \text{peak } \mathbf{2m \cdot a_g = 4 \text{ kN}} \quad (14)$$

3. FEM model

The structure is stored as a regular project file, `verification-time-history-oscillator.json`, with the four runs of section 1 (Dynamics → Time history in the app; `runTimeHistory()` in the core). Shear deformation is off (Settings → Solver), so the element's stiffness is the Euler-Bernoulli k of eq. (1). Each run was solved twice, with the method set to *modal superposition* (3 modes, ζ per mode) and to *Newmark* (average acceleration, HHT $\alpha = 0$, Rayleigh damping at $f_1 = f_2 = 3.3108$ Hz). The program reports its histories every 0.01 s. Displacements of the mass are read from its stored frames (single precision, 7 significant digits); where more digits are needed, from the base shear (double precision): $u = -R_x/k$, and under Newmark with damping $R_x = -k \cdot (u + b_K \cdot \dot{u})$.

4. Results comparison

The reference for modal superposition is the continuous closed form of section 2; the reference for Newmark is its own recurrence (6) — or its closed form (7) — since what is being verified is that the program carries out the method exactly. How far Newmark itself lies from the continuous solution is the subject of section 6.

4.1 TH1 — release, undamped

Quantity	Hand solution [mm]	FEM result [mm]	Rel. difference
Modal superposition against $u_{st} \cdot \cos \omega t$ (8) — from the base shear			
t = 0.00 s	11.554859	11.554858811	0
t = 0.10 s	−5.634688	−5.634688019	1×10^{-15}
t = 0.50 s	−6.473078	−6.473078021	7×10^{-15}
t = 1.00 s	−4.302371	−4.302370527	5×10^{-15}
t = 3.00 s	10.521201	10.521201159	3×10^{-14}
whole run	max $ u_{FEM} - u_{hand} $ / max $ u_{hand} $ over the whole history		3×10^{-14}
Newmark against $u_{st} \cdot \cos \omega t$ (7)–(8) — from the base shear			
t = 0.10 s	−5.615791	−5.615791188	2×10^{-14}
t = 0.50 s	−6.562394	−6.562394293	7×10^{-14}

t = 1.00 s	-4.100848	-4.100848414	3×10^{-13}
t = 3.00 s	10.236443	10.236442789	9×10^{-14}
whole run	$\max u_{\text{FEM}} - u_{\text{hand}} / \max u_{\text{hand}}$ over the whole history		2×10^{-13}

Quantity	Formula	Hand solution	FEM result	Rel. difference
Peak displacement [mm]	u_{st}	11.554859	11.554858811	$< 10^{-15}$
Peak acceleration [m/s ²]	H/m	5.000000	5.000000000	$< 10^{-15}$
f from the free decay, modal [Hz]	$\omega/2\pi$	3.310723	3.310723695	3×10^{-7}
f from the free decay, Newmark [Hz]	$\bar{\omega}/2\pi$	3.307743	3.307743867	3×10^{-7}

The frequency the program measures from the decay of its own run is that run's frequency — ω for modal superposition and $\bar{\omega}$ for Newmark; the measured damping ratio is below 10^{-8} in both.

✓ **Free vibration matches the closed forms to round-off** VERIFIED

4.2 TH2 — step load, $\zeta = 5 \%$

Quantity	Hand solution [mm]	FEM result [mm]	Rel. difference
Modal superposition against (9) — from the base shear			
t = 0.01 s	0.247383	0.247382741	$< 10^{-15}$
t = 0.15 s	21.424966	21.424966191	3×10^{-15}
t = 0.50 s	15.759089	15.759089172	2×10^{-15}
t = 1.00 s	12.784634	12.784633771	4×10^{-15}
t = 3.00 s	11.120599	11.120599238	$< 10^{-15}$
whole run	max u _{FEM} - u _{hand} / max u _{hand} over the whole history		4×10^{-15}
Newmark against the recurrence (6) — from the frames			
t = 0.01 s	0.246734	0.246733805	3×10^{-8}
t = 0.15 s	21.427002	21.427001804	1×10^{-9}
t = 0.50 s	15.815245	15.815244988	2×10^{-8}
t = 1.00 s	12.715477	12.715477496	3×10^{-8}
t = 3.00 s	11.132664	11.132664047	4×10^{-8}
whole run	max u _{FEM} - u _{hand} / max u _{hand} over the whole history		4×10^{-8}

Quantity	Formula	Hand solution	FEM result	Rel. difference
Peak displacement, modal [mm]	eq. (10)	21.428115	21.424966191	1×10^{-4}
Base shear at $t = 0.15$ s, Newmark [kN]	$-k(u + b_K \dot{u})$	-18.555707	-18.555707208	4×10^{-14}

The peak is read at the integration steps (5 ms), which straddle the true maximum at 151.21 ms: hence 10^{-4} there while the history itself agrees to 10^{-15} . The frames hold 7 significant digits, so 10^{-8} is the closest a Newmark displacement can be read.

✓ **The damped step response matches the closed form (modal) and the hand recurrence (Newmark)**

VERIFIED

4.3 TH3 — harmonic load, $\zeta = 5 \%$

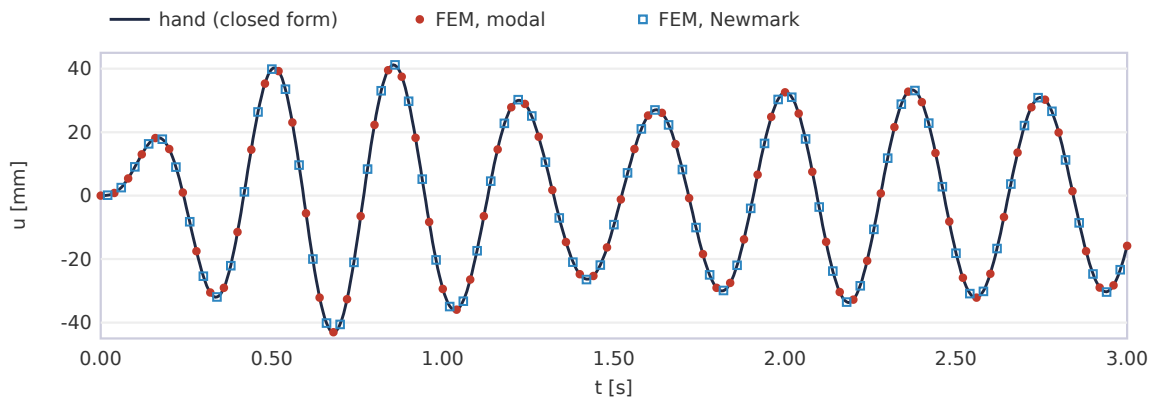


Fig. 2 — TH3: the transient dies out within a few seconds and the steady amplitude $U = 31.335$ mm remains. The first 3 s: closed form (12) as the line, FEM every 0.04 s (the two methods offset by 0.02 s).

Quantity	Hand solution [mm]	FEM result [mm]	Rel. difference
Modal superposition against (12) — from the frames			
t = 0.50 s	39.859542	39.836548269	6×10^{-4}
t = 1.00 s	−29.391044	−29.374089092	6×10^{-4}
t = 2.00 s	32.564933	32.546147704	6×10^{-4}
t = 4.00 s	−11.847236	−11.840401217	6×10^{-4}
t = 8.00 s	25.837088	25.822183117	6×10^{-4}
whole run	max u _{FEM} − u _{hand} / max u _{hand} over the whole history		6×10^{-4}
Newmark against the recurrence (6) — from the frames			
t = 0.50 s	39.845917	39.845917374	4×10^{-10}
t = 1.00 s	−29.476746	−29.476746917	2×10^{-8}
t = 2.00 s	32.528973	32.528974116	2×10^{-8}

t = 4.00 s	-11.844913	-11.844913475	2×10^{-8}
t = 8.00 s	25.876306	25.876305997	2×10^{-8}
whole run	$\max u_{\text{FEM}} - u_{\text{hand}} / \max u_{\text{hand}}$ over the whole history		4×10^{-8}

Quantity	Formula	Hand solution	FEM result	Rel. difference
Steady amplitude U [mm]	eq. (11)	31.334953	31.310066581	8×10^{-4}
Largest u at the output instants of 6–8 s, modal [mm]	eq. (12)	31.328140	31.310066581	6×10^{-4}

Modal superposition integrates each mode exactly for a load that is linear between steps; a sine is not, so here it carries an error of order $(\Omega\Delta t)^2$, 6×10^{-4} at $\Delta t = 5$ ms, falling 4× with every halving of the step (section 6). Newmark reproduces its recurrence to the frames' precision. Its steady amplitude is 0.18 % high because its $\bar{\omega}$ is 0.09 % low, which moves r closer to resonance.

✓ **The harmonic response — transient and steady state — matches the closed form and the recurrence** VERIFIED

4.4 TH4 — ground acceleration step, undamped

Quantity	Hand solution [mm]	FEM result [mm]	Rel. difference
Modal superposition against (13) — from the base shear			
t = 0.05 s	-1.141299	-1.141298837	3×10^{-15}
t = 0.15 s	-4.621419	-4.621418820	2×10^{-15}
t = 0.50 s	-3.605587	-3.605587366	$< 10^{-15}$
t = 2.00 s	-3.981161	-3.981160544	8×10^{-15}
whole run	$\max u_{\text{FEM}} - u_{\text{hand}} / \max u_{\text{hand}}$ over the whole history		9×10^{-15}

Newmark against the recurrence (6) — from the base shear			
t = 0.05 s	-1.139434	-1.139433561	3×10^{-15}
t = 0.15 s	-4.621271	-4.621271417	2×10^{-14}
t = 0.50 s	-3.623451	-3.623450621	1×10^{-13}
t = 2.00 s	-4.039783	-4.039782963	2×10^{-13}
whole run	$\max u_{\text{FEM}} - u_{\text{hand}} / \max u_{\text{hand}}$ over the whole history		3×10^{-13}

Quantity	Formula	Hand solution	FEM result	Rel. difference
Peak relative displacement, modal [mm]	$2a_g/\omega^2$	4.621944	4.621936067	2×10^{-6}
Peak absolute acceleration, modal [m/s ²]	$2a_g$	2.000000	1.999996773	2×10^{-6}

Peak base shear, Newmark [kN]	$2m \cdot a_g$	4.000000	3.999914743	2×10^{-5}
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✓ **Ground excitation: relative motion, absolute acceleration and base shear match** VERIFIED

5. Final results

Final value	Formula	Hand solution	FEM result	Rel. difference
Natural frequency f [Hz]	$\sqrt{k/m}/2\pi$	3.310723	3.310723695	3×10^{-7}
Peak under a sudden 10 kN, $\zeta = 5\%$ [mm]	eq. (9) at the steps	21.424966	21.424966191	3×10^{-15}
Harmonic steady amplitude, $r = 0.8$ [mm]	eq. (12), 6–8 s	31.328140	31.310066581	6×10^{-4}
Peak base shear under $a_g = 1 \text{ m/s}^2$ [kN]	$k \cdot u $, eq. (13)	3.999737	3.999736624	4×10^{-15}

All four from modal superposition. A peak is read where the program has values — at its integration steps or its 0.01 s output instants — so the hand value is the closed form at the same instants. The continuous peaks lie between them: 21.428115 mm (eq. 10), $U = 31.334953$ mm (eq. 11) and $2m \cdot a_g = 4$ kN (eq. 14), within 10^{-3} of the values above. The harmonic amplitude also carries modal superposition's $(\Omega\Delta t)^2$ error on a sine (section 4.3). The frequency is the one the program measures from the free decay of TH1.

✓ **Final: $f = 3.3107$ Hz; step peak 21.428 mm; harmonic amplitude 31.33 mm; base shear 4.00 kN — hand and FEM agree** VERIFIED

6. Time-step convergence

Each run solved again with $\Delta t = 10, 5, 2.5$ and 1.25 ms; the largest difference over the whole history between the FEM displacement of the mass and the *continuous* closed form, relative to the largest displacement:

Δt [ms]	$\omega\Delta t$	TH1 release		TH2 step, $\zeta = 5\%$		TH3 harmonic	
		Newmark	modal	Newmark	modal	Newmark	modal
10	0.2080	2×10^{-1}	4×10^{-8}	1×10^{-2}	4×10^{-8}	2×10^{-2}	2×10^{-3}
5	0.1040	6×10^{-2}	4×10^{-8}	4×10^{-3}	4×10^{-8}	6×10^{-3}	6×10^{-4}
2.5	0.0520	1×10^{-2}	4×10^{-8}	9×10^{-4}	4×10^{-8}	1×10^{-3}	1×10^{-4}
1.25	0.0260	3×10^{-3}	4×10^{-8}	2×10^{-4}	4×10^{-8}	3×10^{-4}	4×10^{-5}

Newmark's error is its period elongation: a phase lag that grows by $2\pi \cdot (\omega\Delta t)^2/12$ per cycle, so it is largest at the end of a long undamped run (TH1: about 10 periods) and falls $4\times$ per halving of Δt . Modal superposition is exact for TH1 and TH2 at any step — the $4 \cdot 10^{-8}$ is the single-precision frame — and converges as Δt^2 on the sine of TH3. In practice, $\Delta t \leq T/20$ ($\omega\Delta t \leq 0.3$) keeps Newmark's phase error below 1 % per cycle.

7. Conclusion

The FEM calculator's linear time-history analysis reproduces the closed-form response of a single-degree-of-freedom oscillator — free vibration, a damped step load, a harmonic load from rest with its transient, and a ground-acceleration step — with both integration methods. Modal superposition equals the continuous solutions to round-off whenever the excitation is piecewise linear between steps, and converges as Δt^2 otherwise. Newmark direct integration reproduces the hand-computed Newmark recurrence to round-off, damping forces in the base shear included, and converges to the continuous solution as Δt^2 , as its period elongation $\bar{\omega} = (2/\Delta t) \cdot \text{atan}(\omega \Delta t/2)$ predicts.

The same comparison runs as part of the automated test suite on every test run: `tests/verificationExamples.test.ts` solves the project file `calc_tests/verification/time-history-oscillator.json` with both methods and asserts the histories, peaks, base shear, measured frequency and time-step convergence against the closed forms and the Newmark recurrence above, recomputed from the section, material and loads stored in the file.