

Verification — Second-Order Analysis of a Cantilever Column

FEM Calculator linear, second-order (P-Δ) and buckling solvers checked against the exact beam-column solution

Structure type	Cantilever column (fixed base, free top), axial load P and horizontal load H at the top
Verification method	Linear: Euler-Bernoulli closed form. Second order: exact solution of the beam-column differential equation. Critical load: Euler, $L_{cr} = 2L$
Quantities checked	Reactions, axial force, moments along the height, tip sway and rotation, axial shortening; critical load factor α_{cr} ; equilibrium on the deformed shape; mesh convergence
Result	Exact match — linear analysis to $\approx 10^{-14}$ (round-off); second order and α_{cr} to $\approx 10^{-6} - 10^{-5}$ with 4 elements, converging as h^4

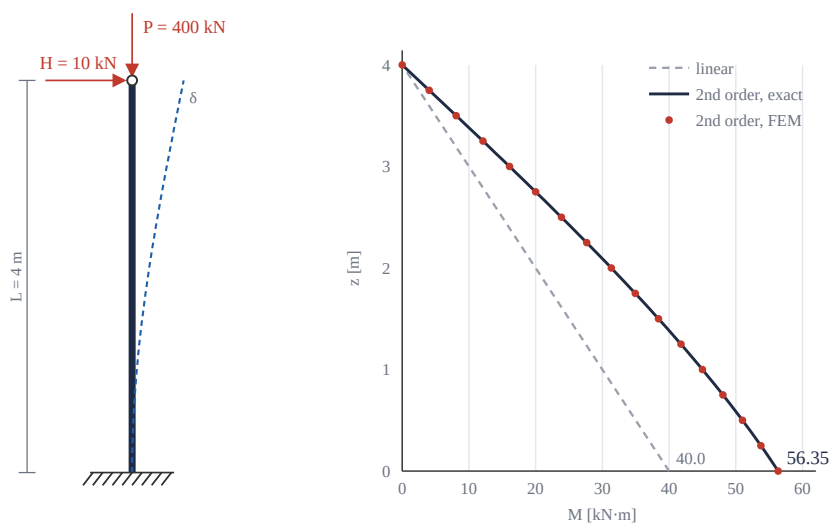


Fig. 1 — Left: the column, its loads and the (exaggerated) deflected shape. Right: bending moment along the height — linear (dashed), exact second-order (solid) and FEM second-order stations (dots).

1. Structure and loading

A single steel column, $L = 4$ m, MSH-H 200×200×8 (EN 10210 hot-finished square hollow section, S235, $E = 210$ GPa, $I = 3\,709.0$ cm⁴, $A = 60.75$ cm², so $EI = 7\,788.9$ kNm² and $EA = 1\,275\,819$ kN). The base is fully fixed, the top is free. At the top it carries a vertical load $P = 400$ kN and a horizontal load $H = 10$ kN. This is the textbook sway case: under P the top's horizontal displacement δ adds a moment $P \cdot \delta$ to the base, which increases δ again — the P-Δ effect that a linear analysis does not see. P was chosen at about one third of the critical load ($\alpha_{cr} \approx 3$), well inside the range where EN 1993-1-1 §5.2.1(3) requires

second-order effects to be taken into account ($\alpha_{cr} < 10$), so the difference between the two analyses is large and easy to read.

2. Hand solution

Data: $L = 4$ m, $P = 400$ kN, $H = 10$ kN, $E = 210$ GPa = $210 \cdot 10^6$ kN/m², $I = 3709.01$ cm⁴ = $3.709015 \cdot 10^{-5}$ m⁴, $A = 60.75$ cm² = $6.075327 \cdot 10^{-3}$ m². Units: kN, m (deflections in mm, rotations in mrad).

$$EI = 210 \cdot 10^6 \text{ kN/m}^2 \cdot 3.709015 \cdot 10^{-5} \text{ m}^4 = \mathbf{7\,788.93 \text{ kN} \cdot \text{m}^2}; \quad EA = 210 \cdot 10^6 \text{ kN/m}^2 \cdot 6.075327 \cdot 10^{-3} \text{ m}^2 = \mathbf{1\,275\,819 \text{ kN}}$$

2.1 Linear (first-order) analysis. Equilibrium on the undeformed shape; the moment is linear in the height and the deflections follow from the cantilever formulas:

$$\text{Reactions: } R_x = H = \mathbf{10 \text{ kN}}, \quad R_z = P = \mathbf{400 \text{ kN}}, \quad N = -P = \mathbf{-400 \text{ kN}}$$

$$\text{Moment: } M_1(z) = H \cdot (L - z) \rightarrow M_{1,\text{base}} = H \cdot L = 10 \text{ kN} \cdot 4 \text{ m} = \mathbf{40 \text{ kN} \cdot \text{m}}; \quad M_1(2 \text{ m}) = 10 \text{ kN} \cdot (4 \text{ m} - 2 \text{ m}) = \mathbf{20 \text{ kN} \cdot \text{m}} \quad (1)$$

$$\text{Tip sway: } \delta_1 = H \cdot L^3 / (3 \cdot EI) = 10 \text{ kN} \cdot (4 \text{ m})^3 / (3 \cdot 7\,788.93 \text{ kN} \cdot \text{m}^2) = \mathbf{27.389295 \text{ mm}} \quad (2)$$

$$\text{Tip rotation: } \theta_1 = H \cdot L^2 / (2 \cdot EI) = 10 \text{ kN} \cdot (4 \text{ m})^2 / (2 \cdot 7\,788.93 \text{ kN} \cdot \text{m}^2) = \mathbf{10.270986 \text{ mrad}}$$

$$\text{Axial shortening: } \Delta_N = P \cdot L / (EA) = 400 \text{ kN} \cdot 4 \text{ m} / 1\,275\,819 \text{ kN} = \mathbf{1.254097 \text{ mm}} \quad (3)$$

2.2 Critical load. A cantilever buckles in the sway mode with buckling length $L_{cr} = 2L$:

$$P_{cr} = \pi^2 \cdot EI / (2L)^2 = \pi^2 \cdot 7\,788.93 \text{ kN} \cdot \text{m}^2 / (2 \cdot 4 \text{ m})^2 = \mathbf{1201.1511 \text{ kN}} \rightarrow \alpha_{cr} = P_{cr} / P = 1201.1511 \text{ kN} / 400 \text{ kN} = \mathbf{3.002878} \quad (4)$$

2.3 Second-order analysis. Equilibrium is written on the deformed shape. With $v(z)$ the lateral deflection and $\delta = v(L)$ the tip sway, the moment at height z is $M(z) = H \cdot (L - z) + P \cdot (\delta - v)$, and $EI \cdot v'' = M$ gives the beam-column equation

$$v'' + k^2 \cdot v = k^2 \cdot [\delta + H \cdot (L - z) / P], \quad k^2 = P / EI, \quad \text{with } v(0) = 0, v'(0) = 0, v(L) = \delta$$

Its solution $v = C_1 \cdot \cos kz + C_2 \cdot \sin kz + \delta + H \cdot (L - z) / P$, with the three conditions fixing C_1 , C_2 and δ , gives the closed forms below ($s = L - z$ is the distance from the top; angles in radians):

$k = \sqrt{P/EI} = \sqrt{(400 \text{ kN} / 7\,788.93 \text{ kN}\cdot\text{m}^2)} = \mathbf{0.226616 \text{ m}^{-1}}$, $kL = 0.226616 \text{ m}^{-1} \cdot 4 \text{ m} = \mathbf{0.906465}$; $\tan kL = 1.277027$, $\cos kL = 0.616533$

Moment: $M_2(s) = H \cdot \sin(ks) / (k \cdot \cos kL)$ (5)

$M_{2,\text{base}} = M_2(L) = H \cdot \tan(kL)/k = 10 \text{ kN} \cdot 1.277027 / 0.226616 \text{ m}^{-1} = \mathbf{56.351975 \text{ kN}\cdot\text{m}}$

$M_2(z = 1 \text{ m}, s = 3 \text{ m}) = 10 \text{ kN} \cdot \sin(0.679849) / (0.226616 \text{ m}^{-1} \cdot 0.616533) = \mathbf{44.996553 \text{ kN}\cdot\text{m}}$

$M_2(z = 2 \text{ m}, s = 2 \text{ m}) = 10 \text{ kN} \cdot \sin(0.453233) / (0.226616 \text{ m}^{-1} \cdot 0.616533) = \mathbf{31.340209 \text{ kN}\cdot\text{m}}$

$M_2(z = 3 \text{ m}, s = 1 \text{ m}) = 10 \text{ kN} \cdot \sin(0.226616) / (0.226616 \text{ m}^{-1} \cdot 0.616533) = \mathbf{16.081266 \text{ kN}\cdot\text{m}}$

Tip sway: $\delta_2 = H/(P \cdot k) \cdot (\tan kL - kL) = 10 \text{ kN} / (400 \text{ kN} \cdot 0.226616 \text{ m}^{-1}) \cdot (1.277027 - 0.906465) = \mathbf{40.879938 \text{ mm}}$ (6)

Tip rotation: $\theta_2 = (H/P) \cdot (1/\cos kL - 1) = (10 \text{ kN} / 400 \text{ kN}) \cdot (1/0.616533 - 1) = \mathbf{15.549344 \text{ mrad}}$ (7)

Check, equilibrium on the deformed shape: $M_{2,\text{base}} = H \cdot L + P \cdot \delta_2 = 10 \text{ kN} \cdot 4 \text{ m} + 400 \text{ kN} \cdot 0.04087994 \text{ m} = \mathbf{56.351975 \text{ kN}\cdot\text{m} \checkmark}$

Amplification: $M_{2,\text{base}}/M_{1,\text{base}} = 56.351975 / 40 = 1.408799$ ($= \tan kL / kL = 1.277027 / 0.906465$); $\delta_2/\delta_1 = 40.879938 / 27.389295 = 1.492552$

The second-order analysis raises the base moment by a factor of 1.409 and the tip sway by 1.493. The shear and the axial force do not change: the horizontal reaction is still H and the vertical one still P, because the loads keep their direction.

3. FEM model

The structure is stored as a regular project file, `verification-second-order-column.json`. The column is split into 4 members of 1 m (5 nodes), so that the geometric stiffness can follow the curved deflected shape. Node 1 is fully fixed; node 5 carries the nodal loads $F_x = 10 \text{ kN}$ and $F_z = -400 \text{ kN}$ in one load case, self-weight off. Shear deformation is off (Settings → Solver), because the hand solution is Euler-Bernoulli. The same model was solved twice with the calculator's core solver (`analyze()`): once with *second-order analysis* off (linear results) and once with it on (P-Δ results, the setting saved in the file). α_{cr} comes from the linear buckling analysis (`bucklingAnalysis()`, the Stability dialog in the app) of the same load case. Values are read at the solver's default 17 stations per member, not interpolated.

4. Results comparison

4.1 Linear analysis (second-order analysis off)

Quantity	Formula	Hand solution	FEM result	Difference
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Horizontal reaction R_x	H	10.000000	9.999999999999801	2×10^{-13}
Vertical reaction R_z	P	400.000000	399.99999999999943	6×10^{-14}
Axial force N	-P	-400.000000	-399.99999999999943	6×10^{-14}
Base moment M_{base}	HL	40.000000	39.999999999999105	9×10^{-13}
Moment at $z = 2$ m	$H \cdot L/2$	20.000000	19.999999999999460	5×10^{-13}
Moment at the top	0	0.000000	0.000000000000114	1×10^{-13}
Tip sway δ_1 [mm]	$HL^3/3EI$	27.389295	27.389294958413540	6×10^{-13}
Tip rotation θ_1 [mrad]	$HL^2/2EI$	10.270986	10.270985609405093	2×10^{-13}
Axial shortening [mm]	PL/EA	1.254097	1.254096627555958	$< 10^{-15}$

✓ **Linear results match to floating-point precision** **VERIFIED**

4.2 Second-order analysis (second-order analysis on) and critical load

Quantity	Formula	Hand solution	FEM result	Rel. difference
Critical load factor α_{cr}	$\pi^2 EI / (4L^2 P)$	3.002878	3.002976418	3×10^{-5}
Base moment $M_{2,\text{base}}$	$H \cdot \tan(kL) / k$	56.351975	56.351886767	2×10^{-6}
Moment at $z = 1$ m	eq. (5), $s = 3$ m	44.996553	44.996471892	2×10^{-6}
Moment at $z = 2$ m	eq. (5), $s = 2$ m	31.340209	31.340147103	2×10^{-6}
Moment at $z = 3$ m	eq. (5), $s = 1$ m	16.081266	16.081233220	2×10^{-6}
Tip sway δ_2 [mm]	eq. (6)	40.879938	40.879716918	5×10^{-6}
Tip rotation θ_2 [mrad]	eq. (7)	15.549344	15.549258494	6×10^{-6}
Amplification M_2/M_1 (base)	$\tan(kL) / kL$	1.408799	1.408797169	2×10^{-6}
Unchanged by second order				
Horizontal reaction R_x	H	10.000000	9.999999999999716	3×10^{-13}
Vertical reaction R_z	P	400.000000	399.99999999999716	3×10^{-13}
Moment at the top	0	0.000000	0.000000000000192	2×10^{-13}
Equilibrium on the deformed shape (FEM against itself)				
M_{base} vs $HL + P \cdot \delta_2, \text{FEM}$	$HL + P \cdot \delta$	56.351887	56.351886767055504	2×10^{-12}

Forces in kN, moments in kN·m. The FEM base moment M_y is positive, its reaction r_y negative; magnitudes are compared. Unlike the linear case, the second-order values are not reproduced to round-off: the element's cubic shape functions only approximate the sine-shaped deflection of a compressed beam, so a small discretisation error remains. It shrinks quickly with the mesh (section 6). The equilibrium row is the exception — the solver's recovery is exact equilibrium on its own deformed shape, so it closes to round-off.

✓ **Second-order results and α_{cr} match the exact solution to better than 4×10^{-5}** VERIFIED

5. Final results

The design-relevant values, by hand and by FEM (the second-order rows as relative difference, 4 members):

Final value	Formula	Hand solution	FEM result	Difference
Base moment, linear $M_{1,base}$ [kN·m]	HL	40.000000	39.999999999999105	9×10^{-13}
Tip sway, linear δ_1 [mm]	$HL^3/3EI$	27.389295	27.389294958413540	6×10^{-13}
Base moment, second order $M_{2,base}$ [kN·m]	$H \cdot \tan(kL)/k$	56.351975	56.351886767	2×10^{-6}
Tip sway, second order δ_2 [mm]	eq. (6)	40.879938	40.879716918	5×10^{-6}
Critical load factor α_{cr}	$\pi^2 EI / (4L^2 P)$	3.002878	3.002976418	3×10^{-5}

✓ **Final: design base moment $M_{2,base} = 56.352$ kN·m (linear 40.000), tip sway 40.880 mm (linear 27.389), $\alpha_{cr} = 3.0029$ — hand and FEM agree** VERIFIED

6. Mesh convergence

The same column was solved with 1, 2, 4 and 8 members. The linear results are exact on any mesh (the cubic element contains the linear solution); the second-order and buckling results converge to the exact values:

Members	α_{cr} FEM	rel. error	δ_2 FEM [mm]	rel. error	$M_{2,base}$ FEM	rel. error
1	3.025466	8×10^{-3}	40.833836	1×10^{-3}	56.333534	3×10^{-4}
2	3.004416	5×10^{-4}	40.876527	8×10^{-5}	56.350611	2×10^{-5}
4	3.002976	3×10^{-5}	40.879717	5×10^{-6}	56.351887	2×10^{-6}
8	3.002885	2×10^{-6}	40.879924	3×10^{-7}	56.351970	1×10^{-7}
exact	3.002878		40.879938		56.351975	

Each halving of the element length cuts the error by about 13–16×, the h^4 convergence expected of the consistent geometric stiffness matrix. Even a single element is within 0.8 % on α_{cr} and 0.11 % on δ_2 ; with 4 elements the errors are of order $10^{-6} - 10^{-5}$.

7. Comparison with the EN 1993-1-1 amplification factor

EN 1993-1-1 §5.2.2(5)B allows the sway effects of a first-order analysis to be amplified by $1/(1 - 1/\alpha_{cr})$ instead of running a second-order analysis (for $\alpha_{cr} \geq 3$). Here $1/(1 - 1/3.0029) = \mathbf{1.4993}$, giving $M_{base} \approx 59.97 \text{ kN}\cdot\text{m}$ — 6.4 % above the exact 56.3520 kN·m. That is the expected outcome: the simplified factor is a safe-side approximation of the exact one (1.4088 for the moment, 1.4926 for the sway), and it confirms that the FEM second-order result sits where theory puts it, between the linear answer and the code's conservative estimate.

8. Conclusion

The FEM calculator's linear analysis reproduces the cantilever formulas to floating-point round-off, and its second-order (P-Δ) analysis reproduces the exact beam-column solution — base moment amplified from 40.00 to 56.3520 kN·m, tip sway from 27.39 to 40.88 mm — with errors of order 10^{-6} on a 4-member column, converging at the theoretical rate. The linear buckling analysis gives the Euler critical load factor of the sway mode to the same accuracy, and the second-order state satisfies equilibrium on the deformed shape exactly.

The same comparison runs as part of the automated test suite on every test run:

`tests/verificationExamples.test.ts` solves the project file `calc_tests/verification/second-order-column.json` and asserts its support reactions, deflections and internal forces against the hand formulas above, recomputed from the section and material stored in the file.