

Verification — Beam with Rotational End Spring

FEM Calculator core solver checked against the closed-form solution of a beam with an elastic end restraint

Structure type	Single-span beam under UDL: pin + roller ($k = 0$), rotational spring + roller ($k = 10\,000\text{ kNm/rad}$), and fixed + roller ($k \rightarrow \infty$)
Verification method	Euler-Bernoulli force method (one redundant, the spring moment), closed form for any k
Quantities checked	Reactions, spring/fixed-end moment, span moments, midspan deflection, end rotations; plus a stiffness sweep from $k = 0$ to $k = 10^8\text{ kNm/rad}$
Result	Exact match — FEM agrees with theory to $\approx 10^{-14}$ (floating-point round-off only)

1. Structure and loading

Three identical beams, $L = 6\text{ m}$, IPE 300 in steel ($E = 210\text{ GPa}$, $I_y = 8\,356\text{ cm}^4$, so $EI = 17\,547.6\text{ kNm}^2$), each carrying a uniformly distributed load $w = 10\text{ kN/m}$ over the full span. They differ only in the rotational restraint at end A; end B is a roller in all three. Beam A is the classical simply supported beam (no spring), Beam B has an elastic rotational spring $k = 10\,000\text{ kNm/rad}$ at A, and Beam C is the rigid limit (A fully fixed against rotation). The spring stiffness was chosen so that kL is of the same order as $3EI$ — the spring then takes roughly half of the fixed-end moment, which is where an error in the spring code path would show most clearly.

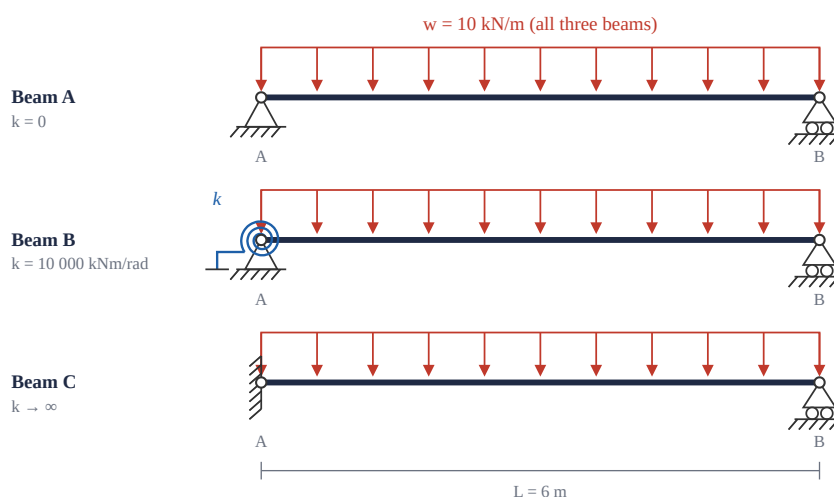


Fig. 1 — The three beams: no rotational restraint, elastic rotational spring at A, and full fixity at A. The span, section and load are identical.

2. Hand solution (force method, Euler-Bernoulli)

Data: $w = 10 \text{ kN/m}$, $L = 6 \text{ m}$, $k = 10\,000 \text{ kNm/rad}$, $EI = E \cdot I_y = 210 \cdot 10^6 \text{ kN/m}^2 \cdot 8\,356 \cdot 10^{-8} \text{ m}^4 = \mathbf{17\,547.6 \text{ kN} \cdot \text{m}^2}$. Units: kN, m, rad (deflections reported in mm, rotations in mrad).

Beam A — pin + roller ($k = 0$).

$$R_A = R_B = w \cdot L / 2 = 10 \text{ kN/m} \cdot 6 \text{ m} / 2 = \mathbf{30.000 \text{ kN}}; \quad M_{\max} = w \cdot L^2 / 8 = 10 \text{ kN/m} \cdot (6 \text{ m})^2 / 8 = \mathbf{45.000 \text{ kN} \cdot \text{m}}$$

$$v(L/2) = 5 \cdot w \cdot L^4 / (384 \cdot EI) = 5 \cdot 10 \text{ kN/m} \cdot (6 \text{ m})^4 / (384 \cdot 17\,547.6 \text{ kN} \cdot \text{m}^2) = \mathbf{9.616700 \text{ mm}}$$

$$\theta_A = w \cdot L^3 / (24 \cdot EI) = 10 \text{ kN/m} \cdot (6 \text{ m})^3 / (24 \cdot 17\,547.6 \text{ kN} \cdot \text{m}^2) = \mathbf{5.128907 \text{ mrad}}$$

Beam B — rotational spring k at A, roller at B. Take the spring moment M_A (hogging) as the redundant. With the spring released, the beam is simply supported and its end A rotates freely by $wL^3/(24EI)$; a moment M_A applied at A rotates it back by $M_A L/(3EI)$. The spring rotates by M_A/k , and compatibility requires the two to be equal:

$$\text{Compatibility at A: } M_A/k = w \cdot L^3 / (24 \cdot EI) - M_A \cdot L / (3 \cdot EI) \rightarrow M_A = (w \cdot L^2 / 8) \cdot k \cdot L / (k \cdot L + 3 \cdot EI) \quad (1)$$

$$k \cdot L / (k \cdot L + 3 \cdot EI) = 10\,000 \text{ kNm/rad} \cdot 6 \text{ m} / (10\,000 \text{ kNm/rad} \cdot 6 \text{ m} + 3 \cdot 17\,547.6 \text{ kN} \cdot \text{m}^2) = 60\,000 / (60\,000 + 52\,642.8) = 0.532657$$

$$M_A = 45 \text{ kN} \cdot \text{m} \cdot 0.532657 = \mathbf{23.969575 \text{ kN} \cdot \text{m}}$$

$$R_A = w \cdot L / 2 + M_A / L = 30 \text{ kN} + 23.969575 \text{ kN} \cdot \text{m} / 6 \text{ m} = \mathbf{33.994929 \text{ kN}}; \quad R_B = w \cdot L / 2 - M_A / L = 30 \text{ kN} - 3.994929 \text{ kN} = \mathbf{26.005071 \text{ kN}}$$

$$\text{Moment: } M(x) = R_A \cdot x - w \cdot x^2 / 2 - M_A \quad (2)$$

$$M(L/2) = w \cdot L^2 / 8 - M_A / 2 = 45 \text{ kN} \cdot \text{m} - 23.969575 \text{ kN} \cdot \text{m} / 2 = \mathbf{33.015213 \text{ kN} \cdot \text{m}}$$

$$M(3.375 \text{ m}) = 33.994929 \text{ kN} \cdot 3.375 \text{ m} - 10 \text{ kN/m} \cdot (3.375 \text{ m})^2 / 2 - 23.969575 \text{ kN} \cdot \text{m} = \mathbf{33.810186 \text{ kN} \cdot \text{m}}$$

$$\text{span peak at } x_0 = R_A / w = 33.994929 \text{ kN} / 10 \text{ kN/m} = 3.3995 \text{ m}; \quad M_{\max} = R_A^2 / (2w) - M_A = (33.994929 \text{ kN})^2 / (2 \cdot 10 \text{ kN/m}) - 23.969575 \text{ kN} \cdot \text{m} = \mathbf{33.8132 \text{ kN} \cdot \text{m}}$$

$$\text{Midspan deflection: } v(L/2) = 5 \cdot w \cdot L^4 / (384 \cdot EI) - M_A \cdot L^2 / (16 \cdot EI) = 9.616700 \text{ mm} - 23.969575 \text{ kN} \cdot \text{m} \cdot (6 \text{ m})^2 / (16 \cdot 17\,547.6 \text{ kN} \cdot \text{m}^2) = \mathbf{6.543257 \text{ mm}} \quad (3)$$

$$\text{End rotations: } \theta_A = M_A / k = 23.969575 \text{ kN} \cdot \text{m} / 10\,000 \text{ kNm/rad} = \mathbf{2.396957 \text{ mrad}}; \quad (4)$$

$$\theta_B = w \cdot L^3 / (24 \cdot EI) - M_A \cdot L / (6 \cdot EI) = 5.128907 \text{ mrad} - 23.969575 \text{ kN} \cdot \text{m} \cdot 6 \text{ m} / (6 \cdot 17\,547.6 \text{ kN} \cdot \text{m}^2) = \mathbf{3.762932 \text{ mrad}}$$

Beam C — fixed at A, roller at B ($k \rightarrow \infty$). Eq. (1) with $k \rightarrow \infty$ gives the propped cantilever:

$$M_A = w \cdot L^2 / 8 = 10 \text{ kN/m} \cdot (6 \text{ m})^2 / 8 = \mathbf{45.000 \text{ kN} \cdot \text{m}}; \quad R_A = 5 \cdot w \cdot L / 8 = 5 \cdot 10 \text{ kN/m} \cdot 6 \text{ m} / 8 = \mathbf{37.500 \text{ kN}}; \quad R_B = 3 \cdot w \cdot L / 8 = 3 \cdot 10 \text{ kN/m} \cdot 6 \text{ m} / 8 = \mathbf{22.500 \text{ kN}}$$

$$\text{Span peak at } x = 5L/8 = 3.75 \text{ m: } M_{\max} = 9 \cdot w \cdot L^2 / 128 = 9 \cdot 10 \text{ kN/m} \cdot (6 \text{ m})^2 / 128 = \mathbf{25.3125 \text{ kN} \cdot \text{m}}$$

$$v(L/2) = w \cdot L^4 / (192 \cdot EI) = 10 \text{ kN/m} \cdot (6 \text{ m})^4 / (192 \cdot 17\,547.6 \text{ kN} \cdot \text{m}^2) = \mathbf{3.846680 \text{ mm}}; \quad \theta_B = w \cdot L^3 / (48 \cdot EI) = 10 \text{ kN/m} \cdot (6 \text{ m})^3 / (48 \cdot 17\,547.6 \text{ kN} \cdot \text{m}^2) = \mathbf{2.564453 \text{ mrad}}$$

3. FEM model

The structure is stored as a regular project file, `verification-rotational-spring-beam.json`, and was solved with the calculator's core solver (`analyze()`) exactly as the app loads it. All three beams sit in one model, 3 m apart and unconnected: one member each ($L = 6 \text{ m}$, IPE 300, S235, $E = 210 \text{ GPa}$, shear deformation off), node A with all translations and torsion restrained, node B a roller (horizontal translation free). Bending rotation r_y at A is free on Beam A, an elastic support spring $k_{ry} = 10\,000 \text{ kNm/rad}$ on Beam B, and rigidly restrained on Beam C. One load case, a full-span global distributed load of -10 kN/m in Z on each member, self-weight off. Result stations are the solver's default 17-point grid; every value below is read directly from that grid, not interpolated.

4. Results comparison

Quantity	Formula	Hand solution	FEM result	Difference
Beam A — pin + roller, no spring ($k = 0$)				
Reaction R_A	$wL/2$	30.000000	29.999999999999989	1×10^{-14}
Reaction R_B	$wL/2$	30.000000	30.000000000000000	0
Midspan moment $M_{\max}$	$wL^2/8$	45.000000	44.999999999999972	3×10^{-14}
Midspan deflection [mm]	$5wL^4/384EI$	9.616700	9.616699719619776	0
End rotation θ_A [mrad]	$wL^3/24EI$	5.128907	5.128906517130548	0
Beam B — rotational spring $k = 10\,000$ kNm/rad at A, roller at B				
Spring moment M_A	eq. (1)	23.969575	23.969574619949075	4×10^{-15}
Reaction R_A	$wL/2 + M_A/L$	33.994929	33.994929103324836	1×10^{-14}
Reaction R_B	$wL/2 - M_A/L$	26.005071	26.005070896675157	4×10^{-15}
Midspan moment $M(L/2)$	$wL^2/8 - M_A/2$	33.015213	33.015212690025436	3×10^{-14}
Moment at $x = 3.375$ m	$R_A x - wx^2/2 - M_A$	33.810186	33.810186103772253	4×10^{-14}
Midspan deflection [mm]	eq. (3)	6.543257	6.543257032592181	0
Spring rotation θ_A [mrad]	M_A/k	2.396957	2.396957461994908	$< 10^{-15}$
End rotation θ_B [mrad]	eq. (4)	3.762932	3.762931989562726	1×10^{-15}
Beam C — fixed at A, roller at B ($k \rightarrow \infty$)				
Fixed-end moment M_A	$wL^2/8$	45.000000	45.000000000000000	0
Reaction R_A	$5wL/8$	37.500000	37.499999999999993	7×10^{-15}
Reaction R_B	$3wL/8$	22.500000	22.500000000000000	0
Span moment $M_{\max}$ ($x = 5L/8$)	$9wL^2/128$	25.312500	25.312499999999972	3×10^{-14}
Midspan deflection [mm]	$wL^4/192EI$	3.846680	3.846679887847911	0
End rotation θ_B [mrad]	$wL^3/48EI$	2.564453	2.564453258565273	$< 10^{-15}$

Forces in kN, moments in kN·m. M_A is reported as a magnitude; the solver gives it as a hogging (negative) M_y and an equal negative r_y reaction. For Beam B the true span peak (33.8132 kN·m at $x_0 = 3.399$ m, eq. 2) lies between two stations; the station at $x = 3.375$ m reproduces eq. (2) exactly, which checks the same moment line.

✓ All quantities match to floating-point precision **VERIFIED**

5. Final results

The design-relevant values of each beam, by hand and by FEM:

Final value	Formula	Hand solution	FEM result	Difference
Beam A — $k = 0$				
Max moment $M_{\max}$ [kN·m]	$wL^2/8$	45.000000	44.999999999999972	3×10^{-14}
Max deflection $v(L/2)$ [mm]	$5wL^4/384EI$	9.616700	9.616699719619776	0
Beam B — $k = 10\,000$ kNm/rad				
Spring moment M_A [kN·m]	eq. (1)	23.969575	23.969574619949075	4×10^{-15}
Midspan moment $M(L/2)$ [kN·m]	$wL^2/8 - M_A/2$	33.015213	33.015212690025436	3×10^{-14}
Max deflection $v(L/2)$ [mm]	eq. (3)	6.543257	6.543257032592181	0
Beam C — $k \rightarrow \infty$				
Fixed-end moment M_A [kN·m]	$wL^2/8$	45.000000	45.000000000000000	0
Max deflection $v(L/2)$ [mm]	$wL^4/192EI$	3.846680	3.846679887847911	0

✓ **Final: spring moment $M_A = 23.9696$ kN·m and midspan deflection 6.5433 mm for the spring beam, between the pin (9.6167 mm) and fixed (3.8467 mm) limits — hand and FEM agree**

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6. Stiffness sweep

To check the spring over its whole range rather than at one value, the stiffness of Beam B was varied from 0 to 10^8 kNm/rad (same model, only k_{ry} changed) and the spring moment and midspan deflection compared with eqs. (1) and (3). The ratio $M_A/(wL^2/8)$ shows the transition from a pin to full fixity.

k [kNm/rad]	$M_A/(wL^2/8)$	M_A hand	M_A FEM	$v(L/2)$ hand [mm]	$v(L/2)$ FEM [mm]	Max diff.
0 (pin)	0.0000	0.000000	0.000000	9.616700	9.616700	7×10^{-15}
1 000	0.1023	4.604146	4.604146	9.026344	9.026344	$< 10^{-15}$
3 000	0.2548	11.466137	11.466137	8.146481	8.146481	2×10^{-15}

10 000	0.5327	23.969575	23.969575	6.543257	6.543257	4×10^{-15}
30 000	0.7737	34.817325	34.817325	5.152330	5.152330	0
100 000	0.9193	41.370256	41.370256	4.312095	4.312095	0
1 000 000	0.9913	44.608613	44.608613	3.896865	3.896865	7×10^{-15}
100 000 000	0.9999	44.996052	44.996052	3.847186	3.847186	7×10^{-15}

✓ Sweep agrees with eqs. (1) and (3) at every stiffness, worst difference 7×10^{-15} **VERIFIED**

7. Conclusion

The FEM calculator reproduces the exact Euler-Bernoulli solution of a beam with an elastic rotational end restraint to within floating-point round-off — reactions, spring moment, span moments, deflection and both end rotations — and converges smoothly onto the two classical limits, the simply supported beam ($k = 0$) and the propped cantilever ($k \rightarrow \infty$). This extends the earlier simply-supported-beam check to support springs, the code path that semi-rigid column bases and partially restrained connections rely on.

The same comparison runs as part of the automated test suite on every test run:

`tests/verificationExamples.test.ts` solves the project file `calc_tests/verification/rotational-spring-beam.json` and asserts its support reactions, deflections and internal forces against the hand formulas above, recomputed from the section and material stored in the file.