Verification — Response Spectrum Analysis of a Two-Mass Cantilever

FEM Calculator modal analysis, EN 1998-1 design spectrum, equivalent static modal forces and their SRSS combination, checked against the exact solution of the two-degree-of-freedom system

Structure type	Steel cantilever, 6 m, with masses of 6 t at mid-height and 4 t at the top; seismic action along X
Verification method	Flexibility matrix from the cantilever formulas, 2×2 eigenproblem in closed form; participation factors and effective masses; EN 1998-1 §3.2.2.5 design spectrum; modal forces, static response per mode and SRSS (§4.3.3.3.2)
Quantities checked	Periods, effective masses, $\Gamma\cdot\varphi$ (all four eigensolvers); $S_d(T)$; modal and SRSS base shear; the exported equivalent static forces; base shear, base moment, moment at mid-height and top displacement per mode and SRSS-combined
Result	**Exact match** — every quantity agrees with the hand solution to round-off (10^{-13} or better)

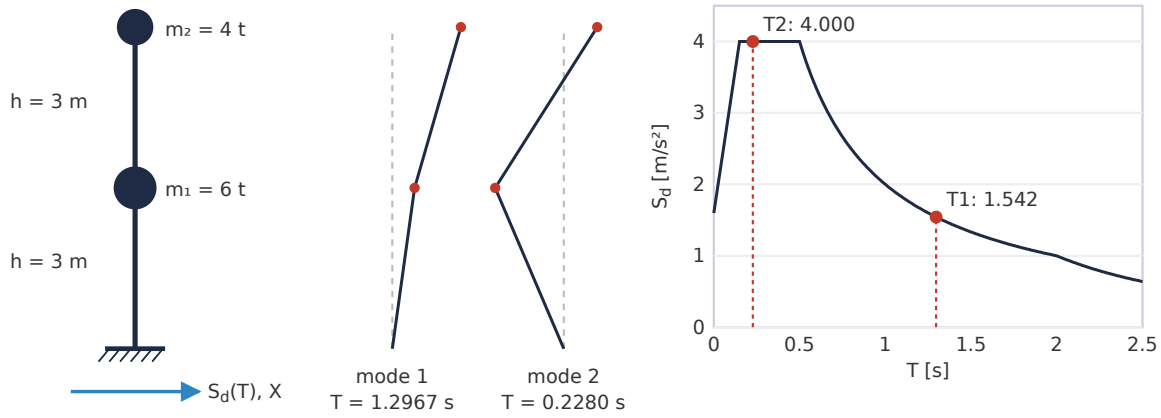

Fig. 1 — Left: the cantilever and its two masses. Middle: the two mode shapes at the nodes. Right: the design spectrum (type 1, ground B, a_g = 2.0 m/s², q = 1.5) with the two modes' ordinates.

1. Structure and loading

A steel column, 2 × 3 m = 6 m, MSH-H 200×200×8 (EN 10210 hot-finished square hollow section, S235, E = 210 GPa, I = 3 709.0 cm⁴), fixed at its base, modelled as two members. Its self weight is left out of the mass; the vibrating masses are m_1 = 6 t at node 2 (z = 3 m) and m_2 = 4 t at node 3 (z = 6 m), entered as a load case of F_z = −58.86 kN and −39.24 kN that is the DYN combination of the modal analysis (m = $|F_z|$/g, g = 9.81 m/s²). The mass acts along X only (Dynamics → Modal analysis, mass directions), so the system has two degrees of freedom: the horizontal displacements of the two masses.

The design spectrum, entered in Dynamics → Response spectrum, is the EN 1998-1 §3.2.2.5 horizontal spectrum with the recommended values:

Parameter	Value	Parameter	Value
Spectrum type, ground type	Type 1, B	S, T_B, T_C, T_D	1.2, 0.15 s, 0.5 s, 2 s
a_{gR}, γ_I (class II)	2 m/s², 1	behaviour factor q, lower bound β	1.5, 0.2

Both modes were exported from the Response spectrum tab as equivalent static load cases (“Seismic mode 1 X”, “Seismic mode 2 X”) and combined in an SRSS combination E_X; the project file holds them as the app left them.

2. Hand solution

Data: h = 3 m, E = 210·10⁶ kN/m², I = 3.709015·10⁻⁵ m⁴, m_1 = 6 t, m_2 = 4 t. Units: kN, m, s, t (1 t = 1 kN·s²/m).

2.1 Flexibility. With the masses only at the nodes and no rotary inertia, the rotations drop out and the structure is described exactly by the lateral flexibility at the two masses — the cantilever formulas, which the cubic element reproduces exactly:

> EI = 210·10⁶ · 3.709015·10⁻⁵ = 7 788.93 kN·m²; h³/EI = 27/7 788.93 = 3.466457643e-03 m/kN
> f_{11} = h³/(3EI) = **1.155485881 mm/kN**; f_{12} = f_{21} = 5h³/(6EI) = **2.888714703 mm/kN**; f_{22} = (2h)³/(3EI) = **9.243887048 mm/kN** (1)

2.2 Eigenproblem. Free vibration u = F·(−M·ü) gives F·M·φ = λ·φ with λ = 1/ω². For 2×2, λ solves λ² − tr(FM)·λ + det(FM) = 0:

> tr = $f_{11}m_1 + f_{22}m_2$ = 4.390846348e-02 s²; det = $m_1m_2(f_{11}f_{22} - f_{12}^2)$ = 5.607620010e-05 s⁴ (2)
> $\lambda_{1,2} = [tr \pm \sqrt{(tr^2 - 4\cdot det)}]/2$ = 4.259186947e-02 s², 1.316594007e-03 s² (3)
> $\omega_1 = 1/\sqrt{\lambda_1}$ = **4.845478251 rad/s**, $T_1 = 2\pi/\omega_1$ = **1.296711074 s**; ω_2 = **27.559673175 rad/s**, T_2 = **0.227984754 s**

2.3 Mode shapes, participation, effective masses. With the top scaled to 1, the first row of (FM − λI)φ = 0 gives the mid-height ordinate; L = Σmφ, M_n = Σmφ², Γ = L/M_n, M_{eff} = L²/M_n:

$\varphi_1 = f_{12}m_2/(\lambda - f_{11}m_1)$: mode 1: $\varphi = (0.324038073, 1)$; mode 2: $\varphi = (-2.057371407, 1)$ (4)

mode 1: L = 5.944228440 t, M_n = 4.630004038 t, Γ_1 = **1.283849515**, $M_{eff,1}$ = **7.631494801 t** (76.3149 %) (5)

mode 2: L = -8.344228440 t, M_n = 29.396662629 t, Γ_2 = **-0.283849515**, $M_{eff,2}$ = **2.368505199 t** (23.6851 %) (6)

check: $M_{eff,1} + M_{eff,2}$ = 10.000000000000 t = $m_1 + m_2$ ✓ (7)

2.4 Design spectrum, EN 1998-1 §3.2.2.5. $a_g = \gamma_I \cdot a_{gR}$ = 2 m/s². T_2 lies on the plateau $T_B \le T \le T_C$, T_1 on the descending branch $T_C \le T \le T_D$:

$S_d(T_2) = a_g \cdot S \cdot 2.5/q = 2.0 \cdot 1.2 \cdot 2.5/1.5$ = **4.000000000 m/s²** (8)

$S_d(T_1) = a_g \cdot S \cdot (2.5/q) \cdot T_C/T_1 = 4.0 \cdot 0.5/1.296711074$ = **1.542363631 m/s²** ≥ $\beta \cdot a_g$ = 0.4 ✓ (9)

2.5 Modal forces and responses. The equivalent static force of mode i at mass j is $F_{ij} = \Gamma_i \cdot S_d(T_i) \cdot m_j \cdot \varphi_{ij}$ (independent of how φ is scaled). Its static response follows from statics and the flexibility (1):

mode 1: F_{11} = **3.849888830 kN**, F_{12} = **7.920651198 kN**; mode 2: F_{21} = **14.015613039 kN**, F_{22} = **-4.541592245 kN** (10)

base shear $V_i = F_{i1} + F_{i2} = \Gamma_i \cdot S_d \cdot L_i = S_d \cdot M_{eff,i}$: V_1 = **11.770540029 kN**, V_2 = **9.474020794 kN** (11)

base moment $M_{b,i} = F_{i1} \cdot h + F_{i2} \cdot 2h$: 59.073573681 kNm, 14.797285646 kNm; at mid-height $M_{h,i} = F_{i2} \cdot h$: 23.761953595, -13.624776736 kNm (12)

top displacement $u_i = f_{21}F_{i1} + f_{22}F_{i2}$ ($= \Gamma_i \cdot S_d/\omega_i^2$): 84.338835495 mm, -1.494858283 mm (13)

2.6 SRSS. T_2/T_1 = 0.1758 ≤ 0.9, so the modes are independent in the sense of §4.3.3.3.2(2) and $E = \sqrt{(E_1^2 + E_2^2)}$ applies; the two modes carry 100 % of the mass (≥ 90 % required by §4.3.3.3.1(3)):

$V = \sqrt{(V_1^2 + V_2^2)}$ = **15.109688368 kN**; M_b = **60.898659837 kNm**; M_h = **27.390965294 kNm**; u_{top} = **84.352082215 mm** (14)

Mode 2 adds little to the base moment (14.80 against 59.07 kNm — its forces act in opposite directions) but 80 % of mode 1's base shear: the classic reason a single-mode (lateral force) estimate understates the shear.

3. FEM model

The structure is stored as a regular project file, `verification-response-spectrum-cantilever.json`: two members of 3 m, node 1 fully fixed, the mass load case LC1 and its DYN combination, the two exported seismic load cases and the SRSS combination E_X (Combinations → Dynamic). Shear deformation is off (Settings → Solver), because the flexibility (1) is Euler-Bernoulli. The modal analysis (`modalAnalysis()`, 2 modes, mass along X) was run with each of the four eigensolvers; the spectrum table (`responseSpectrumTable()`) and the exported forces (`modalCombinedLoad()`) are what the Response spectrum tab shows and creates; the load cases and the SRSS combination were solved by the linear static analysis (`analyze()`). Values are the solver's own, unrounded.

4. Results comparison

4.1 Modal analysis

Quantity	Formula	Hand solution	FEM result	Rel. difference
Eigensolver: subspace iteration				
T_1 [s]	$2\pi/\omega$	1.296711	1.296711074	$< 10^{-15}$
$M_{eff,1}$ [t]	L^2/M_n	7.631495	7.631494801	$< 10^{-15}$
$\Gamma_1 \cdot \varphi_1$ at the top [–]	$\Gamma \cdot \varphi_2$	1.283850	1.283849515	0
$\Gamma_1 \cdot \varphi_1$ at mid-height [–]	$\Gamma \cdot \varphi_1$	0.416016	0.416016123	$< 10^{-15}$
T_2 [s]	$2\pi/\omega$	0.227985	0.227984754	$< 10^{-15}$
$M_{eff,2}$ [t]	L^2/M_n	2.368505	2.368505199	0
$\Gamma_2 \cdot \varphi_2$ at the top [–]	$\Gamma \cdot \varphi_2$	−0.283850	−0.283849515	0
$\Gamma_2 \cdot \varphi_2$ at mid-height [–]	$\Gamma \cdot \varphi_1$	0.583984	0.583983877	0
Eigensolver: Lanczos				
T_1 [s]	$2\pi/\omega$	1.296711	1.296711074	$< 10^{-15}$
$M_{eff,1}$ [t]	L^2/M_n	7.631495	7.631494801	$< 10^{-15}$
$\Gamma_1 \cdot \varphi_1$ at the top [–]	$\Gamma \cdot \varphi_2$	1.283850	1.283849515	0
$\Gamma_1 \cdot \varphi_1$ at mid-height [–]	$\Gamma \cdot \varphi_1$	0.416016	0.416016123	$< 10^{-15}$
T_2 [s]	$2\pi/\omega$	0.227985	0.227984754	$< 10^{-15}$

$M_{eff,2}$ [t]	L^2/M_n	2.368505	2.368505199	$< 10^{-15}$
$\Gamma_2 \cdot \varphi_2$ at the top [–]	$\Gamma \cdot \varphi_2$	−0.283850	−0.283849515	0
$\Gamma_2 \cdot \varphi_2$ at mid-height [–]	$\Gamma \cdot \varphi_1$	0.583984	0.583983877	$< 10^{-15}$
Eigensolver: dense				
T_1 [s]	$2\pi/\omega$	1.296711	1.296711074	$< 10^{-15}$
$M_{eff,1}$ [t]	L^2/M_n	7.631495	7.631494801	0
$\Gamma_1 \cdot \varphi_1$ at the top [–]	$\Gamma \cdot \varphi_2$	1.283850	1.283849515	$< 10^{-15}$
$\Gamma_1 \cdot \varphi_1$ at mid-height [–]	$\Gamma \cdot \varphi_1$	0.416016	0.416016123	$< 10^{-15}$
T_2 [s]	$2\pi/\omega$	0.227985	0.227984754	$< 10^{-15}$
$M_{eff,2}$ [t]	L^2/M_n	2.368505	2.368505199	6×10^{-15}
$\Gamma_2 \cdot \varphi_2$ at the top [–]	$\Gamma \cdot \varphi_2$	−0.283850	−0.283849515	$< 10^{-15}$
$\Gamma_2 \cdot \varphi_2$ at mid-height [–]	$\Gamma \cdot \varphi_1$	0.583984	0.583983877	4×10^{-15}
Eigensolver: Sturm bisection				
T_1 [s]	$2\pi/\omega$	1.296711	1.296711074	2×10^{-15}
$M_{eff,1}$ [t]	L^2/M_n	7.631495	7.631494801	4×10^{-14}
$\Gamma_1 \cdot \varphi_1$ at the top [–]	$\Gamma \cdot \varphi_2$	1.283850	1.283849515	5×10^{-15}
$\Gamma_1 \cdot \varphi_1$ at mid-height [–]	$\Gamma \cdot \varphi_1$	0.416016	0.416016123	1×10^{-13}
T_2 [s]	$2\pi/\omega$	0.227985	0.227984754	$< 10^{-15}$
$M_{eff,2}$ [t]	L^2/M_n	2.368505	2.368505199	1×10^{-13}
$\Gamma_2 \cdot \varphi_2$ at the top [–]	$\Gamma \cdot \varphi_2$	−0.283850	−0.283849515	2×10^{-14}
$\Gamma_2 \cdot \varphi_2$ at mid-height [–]	$\Gamma \cdot \varphi_1$	0.583984	0.583983877	7×10^{-14}

$\Gamma \cdot \varphi$ is compared rather than Γ and φ separately, because the program scales each shape to max $|u| = 1$ and the sign of an eigenvector is arbitrary; $\Gamma \cdot \varphi$ — what the forces are made of — is neither.

✓ Periods, effective masses and participation match the closed form to round-off, with all four eigensolvers VERIFIED

4.2 Design spectrum and modal base shears (Response spectrum table)

Quantity	Formula	Hand solution	FEM result	Rel. difference

$S_d(T_1)$ [m/s²]	(9)	1.542364	1.542363631	$< 10^{-15}$
$V_1 = S_d \cdot M_{eff}$ [kN]	(11)	11.770540	11.770540029	$< 10^{-15}$
$S_d(T_2)$ [m/s²]	(8)	4.000000	4.000000000	0
$V_2 = S_d \cdot M_{eff}$ [kN]	(11)	9.474021	9.474020794	6×10^{-15}
$V = \sqrt{(V_1^2 + V_2^2)}$ [kN]	(14)	15.109688	15.109688368	2×10^{-15}

4.3 Exported equivalent static forces

Quantity	Formula	Hand solution	FEM result	Rel. difference
Load case “Seismic mode 1 X”				
F at node 2 (z = 3 m) [kN]	$\Gamma \cdot S_d \cdot m_1 \cdot \varphi_1$	3.849889	3.849888830	0
F at node 3 (z = 6 m) [kN]	$\Gamma \cdot S_d \cdot m_2 \cdot \varphi_2$	7.920651	7.920651198	$< 10^{-15}$
Load case “Seismic mode 2 X”				
F at node 2 (z = 3 m) [kN]	$\Gamma \cdot S_d \cdot m_1 \cdot \varphi_1$	14.015613	14.015613039	4×10^{-15}
F at node 3 (z = 6 m) [kN]	$\Gamma \cdot S_d \cdot m_2 \cdot \varphi_2$	−4.541592	−4.541592245	$< 10^{-15}$

✓ **Spectrum ordinates, base shears and the exported forces match** VERIFIED

4.4 Static results of the seismic load cases and of their SRSS combination

Quantity	Formula	Hand solution	FEM result	Rel. difference
Mode 1 load case				
Base shear [kN]	ΣF	11.770540	11.770540029	2×10^{-15}
Base moment [kNm]	$\Sigma F \cdot z$	59.073574	59.073573681	$< 10^{-15}$
Moment at mid-height [kNm]	$F_2 \cdot h$	23.761954	23.761953595	$< 10^{-15}$
Top displacement [mm]	$f_{21}F_1 + f_{22}F_2$	84.338835	84.338835495	$< 10^{-15}$
Mode 2 load case				
Base shear [kN]	ΣF	9.474021	9.474020794	7×10^{-15}
Base moment [kNm]	$\Sigma F \cdot z$	14.797286	14.797285646	1×10^{-14}
Moment at mid-height [kNm]	$F_2 \cdot h$	13.624777	13.624776736	1×10^{-15}
Top displacement [mm]	$f_{21}F_1 + f_{22}F_2$	−1.494858	−1.494858283	1×10^{-13}

SRSS combination E_X				
Base shear [kN]	$\sqrt{\Sigma V^2}$	15.109688	15.109688368	2×10^{-15}
Base moment [kNm]	$\sqrt{\Sigma M^2}$	60.898660	60.898659837	$< 10^{-15}$
Moment at mid-height [kNm]	$\sqrt{\Sigma M^2}$	27.390965	27.390965294	$< 10^{-15}$
Top displacement [mm]	$\sqrt{\Sigma u^2}$	84.352082	84.352082215	$< 10^{-15}$

Base moment and moments are compared by magnitude: the program reports a reaction with the sign of a force on the structure and the SRSS combination takes its sign from the leading case (mode 1), both of which are conventions rather than results.

✓ Internal forces, reactions and displacements per mode and combined match VERIFIED

5. Final results

Final value	Formula	Hand solution	FEM result	Rel. difference
Fundamental period T_1 [s]	$2\pi\sqrt{\lambda_1}$	1.296711	1.296711074	$< 10^{-15}$
Effective mass of mode 1 [%]	$M_{eff,1}/\Sigma m$	76.3149	76.314948	0
Seismic base shear E_X [kN]	$\sqrt{(V_1^2 + V_2^2)}$	15.109688	15.109688368	2×10^{-15}
Seismic base moment E_X [kNm]	$\sqrt{(M_1^2 + M_2^2)}$	60.898660	60.898659837	$< 10^{-15}$
Top displacement d_e [mm]	$\sqrt{(u_1^2 + u_2^2)}$	84.352082	84.352082215	$< 10^{-15}$

✓ Final: T_1 = 1.2967 s; V = 15.110 kN; M_b = 60.899 kNm; d_e = 84.352 mm — hand and FEM agree VERIFIED

The displacement is d_e, from the design spectrum, which is reduced by q. EN 1998-1 §4.3.4 takes the design displacement as $d_s = q_d \cdot d_e = 1.5 \cdot 84.352 = 126.528$ mm (q_d = q unless specified); the program reports d_e and leaves that factor to the designer, as it does the accidental torsional effects of §4.3.2, which a single cantilever does not have.

6. Conclusion

The FEM calculator's response-spectrum chain — modal analysis, EN 1998-1 design spectrum, modal base shears, exported equivalent static forces, static analysis of each mode and SRSS combination — reproduces the exact solution of a two-degree-of-freedom cantilever to round-off at every step, with all four eigensolvers. The second mode, on the plateau of the spectrum, carries 80 % of the first mode's base shear, and the SRSS combination of the two is what the program reports for the seismic action.

The same comparison runs as part of the automated test suite on every test run: `tests/verificationExamples.test.ts` solves the project file `calc_tests/verification/response-spectrum-cantilever.json` and asserts the periods, effective masses, participation, spectrum ordinates, base shears, exported forces and the static results per mode and SRSS-combined against the hand solution above, recomputed from the section, material and masses stored in the file.