

Verification — Euler Buckling of Columns with Four Support Conditions

FEM Calculator linear buckling analysis (critical load factors α_{cr} , mode shapes, buckling lengths) checked against Euler's closed-form solutions

Structure type	Four axially loaded steel columns side by side: fixed–free, pinned–pinned, fixed–pinned, fixed–fixed
Verification method	Euler: $P_{cr} = \pi^2 \cdot EI / L_{cr}^2$, from the characteristic equation of each support case ($\tan kL = kL$ for fixed–pinned, solved by Newton's method); exact buckled shapes
Quantities checked	The eight lowest critical load factors of the whole model and their order; repeated pairs ($I_y = I_z$); first and second factors of each column; mode shape; buckling lengths; the EN 1993-1-1 §5.2.1(3) check; mesh convergence from 1 to 1 000 members
Result	Match — first factors within $10^{-6} - 2 \cdot 10^{-4}$ with 10 members (second factors within 10^{-3}), always from above, converging as h^4 to the solver's 10^{-6} tolerance and holding there on fine meshes; mode shapes exact at the nodes

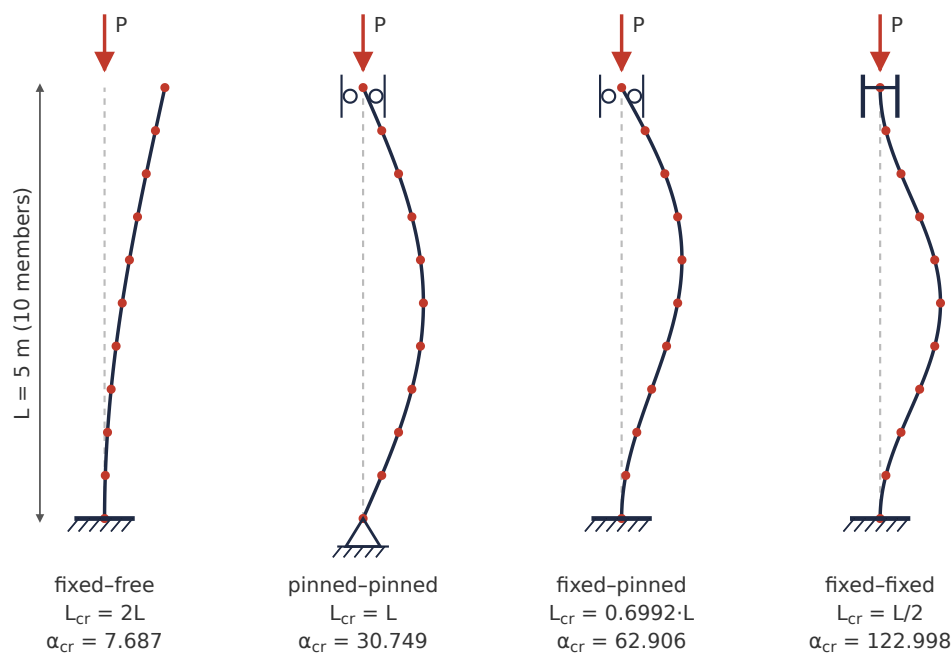


Fig. 1 — The four columns, their supports and the load $P = 100$ kN. Line: the exact first buckled shape; dots: the FEM mode shape at the nodes (10 members per column).

1. Structure and loading

Four steel columns, each $L = 5$ m, MSH-H 200×200×8 (EN 10210 hot-finished square hollow section, S235, $E = 210$ GPa, $I_y = I_z = 3\,709.0$ cm⁴, $A = 60.75$ cm²), stand 3 m apart and are not connected to each

other. Each carries an axial load $P = 100 \text{ kN}$ at its top. They differ only in their supports, which give the four classical Euler cases:

Column	Base ($z = 0$)	Top ($z = 5 \text{ m}$)	Euler case
1	fixed	free	fixed–free (cantilever)
2	pinned: u_x, u_y, u_z and torsion r_z held	u_x, u_y held	pinned–pinned
3	fixed	u_x, u_y held	fixed–pinned
4	fixed	everything held except u_z (sliding clamp)	fixed–fixed

The top of every column is free to move axially, so each carries exactly $N = -P$. The square section bends alike about both axes, so every critical load comes twice — once in each plane. That is deliberate: a repeated eigenvalue is where an eigenvalue solver most easily skips a mode.

2. Hand solution

Data: $L = 5 \text{ m}$, $P = 100 \text{ kN}$, $E = 210 \cdot 10^6 \text{ kN/m}^2$, $I = 3\,709.01 \text{ cm}^4 = 3.709015 \cdot 10^{-5} \text{ m}^4$. Units: kN , m . The critical load factor is $\alpha_{\text{cr}} = P_{\text{cr}}/P$.

$$EI = 210 \cdot 10^6 \text{ kN/m}^2 \cdot 3.709015 \cdot 10^{-5} \text{ m}^4 = \mathbf{7\,788.93 \text{ kN} \cdot \text{m}^2}$$

$$P_E = \pi^2 \cdot EI/L^2 = \pi^2 \cdot 7\,788.93 \text{ kN} \cdot \text{m}^2 / (5 \text{ m})^2 = \mathbf{3\,074.9468 \text{ kN}} \rightarrow \alpha_E = P_E/P = \mathbf{30.749468} \quad (1)$$

2.1 The equation. With $v(z)$ the lateral deflection and $k^2 = P/EI$, equilibrium of the deflected column, $EI \cdot v'' = -P \cdot v$ + (end moment and shear terms), gives $v'''' + k^2 \cdot v'' = 0$ with the general solution $v = C_1 \cdot \cos kz + C_2 \cdot \sin kz + C_3 \cdot z + C_4$. Each support case is four homogeneous end conditions on $C_1 \dots C_4$; a non-zero shape exists only where their determinant vanishes, which fixes kL and so $P_{\text{cr}} = (kL)^2 \cdot EI/L^2$.

$$\text{Fixed–free: } v(0) = v'(0) = 0, M(L) = V(L) = 0 \rightarrow \cos kL = 0 \rightarrow kL = \pi/2, 3\pi/2, \dots; v = 1 - \cos(\pi z/2L) \quad (2)$$

$$\text{Pinned–pinned: } v = v'' = 0 \text{ at both ends} \rightarrow \sin kL = 0 \rightarrow kL = \pi, 2\pi, \dots; v = \sin(\pi z/L) \quad (3)$$

$$\text{Fixed–pinned: } v(0) = v'(0) = 0, v(L) = v''(L) = 0 \rightarrow \tan kL = kL; v = (L - z) - L \cdot \cos kz + \sin(kz)/k \quad (4)$$

$$\text{Fixed–fixed: } v = v' = 0 \text{ at both ends} \rightarrow kL = 2\pi \text{ (symmetric), then } \tan(kL/2) = kL/2, \text{ i.e. } kL = 2 \cdot x_1; v = 1 - \cos(2\pi z/L) \quad (5)$$

2.2 The roots of $\tan x = x$. Newton's method on $f(x) = \tan x - x$, $f'(x) = \tan^2 x$, $x \leftarrow x - (\tan x - x)/\tan^2 x$:

first root: $x = 4.49 \rightarrow 4.493464334 \rightarrow 4.493409472 \rightarrow 4.493409457909 \Rightarrow \mathbf{x_1 = 4.493409458}$ (6)

second root: $x = 7.72 \rightarrow 7.725468684 \rightarrow 7.725252206 \rightarrow 7.725251836939 \Rightarrow \mathbf{x_2 = 7.725251837}$

buckling length of the fixed–pinned column: $L_{cr} = \pi/x_1 \cdot L = 0.699156 \cdot 5 \text{ m} = \mathbf{3.495778 \text{ m}}$ (the textbook $0.7 \cdot L$)

2.3 The critical load factors. $\alpha_{cr} = (kL/\pi)^2 \cdot \alpha_E$:

$$\text{Fixed–free: } \alpha_1 = \alpha_E/4 = 30.749468/4 = \mathbf{7.687367}; \alpha_2 = 9 \cdot \alpha_E/4 = \mathbf{69.186302} \quad (7)$$

$$\text{Pinned–pinned: } \alpha_1 = \alpha_E = \mathbf{30.749468}; \alpha_2 = 4 \cdot \alpha_E = \mathbf{122.997871} \quad (8)$$

$$\text{Fixed–pinned: } \alpha_1 = (x_1/\pi)^2 \cdot \alpha_E = 2.045748516 \cdot 30.749468 = \mathbf{62.905678}; \alpha_2 = (x_2/\pi)^2 \cdot \alpha_E = \mathbf{185.935857} \quad (9)$$

$$\text{Fixed–fixed: } \alpha_1 = 4 \cdot \alpha_E = \mathbf{122.997871}; \alpha_2 = (2x_1/\pi)^2 \cdot \alpha_E = \mathbf{251.622712} \quad (10)$$

Sorted, the lowest eight factors of the whole model are: column 1 mode 1 (7.687), column 2 mode 1 (30.749), column 3 mode 1 (62.906) and column 1 mode 2 (69.186) — each twice, once per bending plane. The next ones, at 122.998, are four coincident factors (column 2 mode 2 and column 4 mode 1, both planes).

2.4 EN 1993-1-1 §5.2.1(3). The governing factor is $\alpha_{cr} = 7.687$, between 3 and 10: second-order effects must be taken into account, and the amplification of §5.2.2(5)B may be used.

3. FEM model

The structure is stored as a regular project file, `verification-euler-buckling-columns.json`. Each column is split into 10 members of 0.5 m (11 nodes); the supports are as in section 1, and one load case carries the four nodal loads $F_z = -100 \text{ kN}$, self-weight off. Shear deformation is off (Settings → Solver), because Euler's solution ignores it. The structure was solved linearly (`analyze()`), and the linear buckling analysis (`bucklingAnalysis()` , the Stability dialog in the app) was run on that load case. It finds the factors with Sturm-sequence counts on $K + \alpha \cdot K_G$ — the number of negative pivots of its LDL^T factorisation equals the number of critical factors below α — to a relative tolerance of 10^{-6} . For the mesh study (section 6) each column was also solved on its own, with the same supports, meshed into 1 to 1 000 members.

4. Results comparison

4.1 The eight lowest factors of the model, in the order the solver returns them

Mode	Column and mode	Hand solution	FEM result	Difference
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1	fixed–free, mode 1	7.687367	7.687374	$+9 \times 10^{-7}$
2	fixed–free, mode 1	7.687367	7.687374	$+9 \times 10^{-7}$
3	pinned–pinned, mode 1	30.749468	30.749893	$+1 \times 10^{-5}$
4	pinned–pinned, mode 1	30.749468	30.749893	$+1 \times 10^{-5}$
5	fixed–pinned, mode 1	62.905678	62.909210	$+6 \times 10^{-5}$
6	fixed–pinned, mode 1	62.905678	62.909210	$+6 \times 10^{-5}$
7	fixed–free, mode 2	69.186302	69.191010	$+7 \times 10^{-5}$
8	fixed–free, mode 2	69.186302	69.191010	$+7 \times 10^{-5}$

Every factor comes as a pair, equal to 10^{-6} — one in each bending plane — and none is skipped. Every FEM value lies above Euler: the cubic element with the consistent geometric stiffness can only overestimate a buckling load, by an amount that grows with the number of half-waves per member (column 1's second mode bends in three quarter-waves over the same 10 members).

✓ All eight factors, their order and the repeated pairs match Euler VERIFIED

4.2 Each column on its own (10 members), first and second factor

Quantity	Formula	Hand solution	FEM result	Rel. difference
Column 1: fixed–free				
$\alpha_{cr,1}$	$(k_1 L/L)^2 \cdot EI/P$	7.687367	7.687374	9×10^{-7}
$\alpha_{cr,2}$	$(k_2 L/L)^2 \cdot EI/P$	69.186302	69.191010	7×10^{-5}
Column 2: pinned–pinned				
$\alpha_{cr,1}$	$(k_1 L/L)^2 \cdot EI/P$	30.749468	30.749893	1×10^{-5}
$\alpha_{cr,2}$	$(k_2 L/L)^2 \cdot EI/P$	122.997871	123.023987	2×10^{-4}
Column 3: fixed–pinned				
$\alpha_{cr,1}$	$(k_1 L/L)^2 \cdot EI/P$	62.905678	62.909210	6×10^{-5}
$\alpha_{cr,2}$	$(k_2 L/L)^2 \cdot EI/P$	185.935857	186.024963	5×10^{-4}
Column 4: fixed–fixed				
$\alpha_{cr,1}$	$(k_1 L/L)^2 \cdot EI/P$	122.997871	123.023987	2×10^{-4}
$\alpha_{cr,2}$	$(k_2 L/L)^2 \cdot EI/P$	251.622712	251.840210	9×10^{-4}

Second factors carry a larger discretisation error, as they bend in more waves. The fixed–fixed column's first mode is a full wave of length $L/2$ between inflection points, so on 10 members it has the error of a pinned–pinned column on 5.

4.3 Mode shape of the pinned–pinned column

The lateral displacement of column 2 in mode 3 of the model, scaled to 1 at mid-height, against $\sin(\pi z/L)$ (eq. 3). The pair's modes are free to mix the two planes, so the magnitude $\sqrt{(u_x^2 + u_y^2)}$ is compared:

z [m]	$\sin(\pi z/L)$	FEM	Difference
0.0	0.000000000000	0.000000000000	0
0.5	0.309016994375	0.309016994375	-4×10^{-15}
1.0	0.587785252292	0.587785252292	-7×10^{-15}
1.5	0.809016994375	0.809016994375	-6×10^{-15}
2.0	0.951056516295	0.951056516295	-3×10^{-15}
2.5	1.000000000000	1.000000000000	0
3.0	0.951056516295	0.951056516295	$+2 \times 10^{-15}$
3.5	0.809016994375	0.809016994375	$+3 \times 10^{-15}$
4.0	0.587785252292	0.587785252292	$+3 \times 10^{-15}$
4.5	0.309016994375	0.309016994375	$+2 \times 10^{-15}$
5.0	0.000000000000	0.000000000000	$< 10^{-15}$

The nodal values are the sine to round-off; the cubic element interpolates between the nodes, but the nodal values of its buckling mode are exact.

✓ The buckled shape is the half sine **VERIFIED**

5. Final results

Final value	Formula	Hand solution	FEM result	Rel. difference
α_{cr} of the structure	$\pi^2 EI / (2L)^2 / P$	7.687367	7.687374	9×10^{-7}
L_{cr} fixed–free [m]	$2 \cdot L$	10.000000	9.999995	5×10^{-7}
L_{cr} pinned–pinned [m]	L	5.000000	4.999965	7×10^{-6}
L_{cr} fixed–pinned [m]	$\pi/x_1 \cdot L$	3.495778	3.495680	3×10^{-5}
L_{cr} fixed–fixed [m]	$L/2$	2.500000	2.499735	1×10^{-4}

$L_{cr} = \pi \cdot \sqrt{(EI / (\alpha_{cr} \cdot |N|))}$ is the buckling length the Stability dialog suggests for each member from its factor and axial force. The stability check that runs after every solve reports $\alpha_{cr} = 7.6874$, "second-order effects significant" ($3 \leq \alpha_{cr} < 10$), as §5.2.1(3) gives in section 2.4.

✓ Final: $\alpha_{cr} = 7.6874$ (hand 7.6874); buckling lengths $2L$, L , $0.699L$ and $L/2$ — hand and FEM agree

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6. Mesh convergence

Each column solved alone, meshed into n members; first critical load factor and its relative difference from Euler:

n	fixed-free		pinned-pinned		fixed-pinned		fixed-fixed	
1	7.745197	$+8 \times 10^{-3}$	37.386856	$+2 \times 10^{-1}$	93.467194	$+5 \times 10^{-1}$	—	
2	7.691303	$+5 \times 10^{-4}$	30.980789	$+8 \times 10^{-3}$	64.519745	$+3 \times 10^{-2}$	124.622864	$+1 \times 10^{-2}$
4	7.687618	$+3 \times 10^{-5}$	30.765213	$+5 \times 10^{-4}$	63.034943	$+2 \times 10^{-3}$	123.923157	$+8 \times 10^{-3}$
8	7.687382	$+2 \times 10^{-6}$	30.750473	$+3 \times 10^{-5}$	62.914215	$+1 \times 10^{-4}$	123.060852	$+5 \times 10^{-4}$
10	7.687374	$+9 \times 10^{-7}$	30.749893	$+1 \times 10^{-5}$	62.909210	$+6 \times 10^{-5}$	123.023987	$+2 \times 10^{-4}$
16	7.687366	-6×10^{-8}	30.749527	$+2 \times 10^{-6}$	62.906219	$+9 \times 10^{-6}$	123.001892	$+3 \times 10^{-5}$
32	7.687366	-6×10^{-8}	30.749466	-6×10^{-8}	62.905731	$+8 \times 10^{-7}$	122.998108	$+2 \times 10^{-6}$
100	7.687366	-6×10^{-8}	30.749466	-6×10^{-8}	62.905670	-1×10^{-7}	122.997864	-6×10^{-8}
300	7.687366	-6×10^{-8}	30.749466	-6×10^{-8}	62.905670	-1×10^{-7}	122.997864	-6×10^{-8}
1000	7.687138	-3×10^{-5}	30.749435	-1×10^{-6}	62.905670	-1×10^{-7}	122.997864	-6×10^{-8}
Euler	7.687367		30.749468		62.905678		122.997871	

On coarse meshes every value is above Euler, and every doubling of the mesh cuts the error by about $16\times$ (pinned-pinned: $14.7\times$ from 2 to 4 members, $15.7\times$ from 4 to 8) — the h^4 rate of the cubic element. A single member already gives the cantilever within 0.8 %; the fixed-fixed column with one member has no free lateral degree of freedom and so no mode. By 100 members all four are within the solver's 10^{-6} tolerance on α (the $-6 \cdot 10^{-8}$ is that tolerance), and they stay there to 300. At 1 000 members (5 mm elements on a 5 m column) round-off begins to show on the cantilever: the stiffness of one element is then some 10^9 times the lateral stiffness of the whole column, and double precision carries the difference to a few significant digits fewer. The counts themselves stay exact: near a critical factor the pivot about to change sign is tiny next to its own diagonal, and the solver takes it by its sign rather than discarding it as a zero-stiffness mechanism.

7. Conclusion

The FEM calculator's linear buckling analysis reproduces Euler's critical loads for all four classical support cases — first and second modes, both bending planes, in the correct order with no repeated factor skipped — and the buckled shapes and buckling lengths that go with them. With 10 members per column the governing factor is within 10^{-6} of the hand value (7.687374 against 7.687367), the other first factors within $2 \cdot 10^{-4}$ and the second ones within 10^{-3} , every one from above, and the error falls with the fourth power of the element length down to the solver's tolerance.

The same comparison runs as part of the automated test suite on every test run: `tests/verificationExamples.test.ts` solves the project file `calc_tests/verification/euler-buckling-columns.json` and asserts the critical load factors, their order and pairs, the mode shape, the buckling lengths, the §5.2.1(3) verdict and the mesh convergence against the closed forms above, recomputed from the section and material stored in the file.