Verification — Natural Frequencies of a Simply Supported Beam

FEM Calculator modal (eigenvalue) analysis checked against the classical beam frequencies and the exact solution of the lumped-mass model

Structure type	Simply supported steel beam (pin + roller), mass from its self weight
Verification method	Euler-Bernoulli beam: $f_n = n^2\pi/(2L^2)\cdot\sqrt{(EI/\bar{m})}$. Lumped-mass model solved in closed form (sine modes, aliased flexibility series). Axial rod: $f = \sqrt{(E/\rho)}/(4L)$
Quantities checked	Lumped masses; vertical frequencies of modes 1–3; weak-axis and axial modes and their order; effective modal masses; both eigensolvers (subspace, Lanczos); mesh convergence
Result	**Exact match** — frequencies equal the lumped-mass closed form to $\approx 10^{-12}$ and the continuous beam to $4\cdot10^{-7}$ – $4\cdot10^{-5}$ with 20 members, converging as h^4

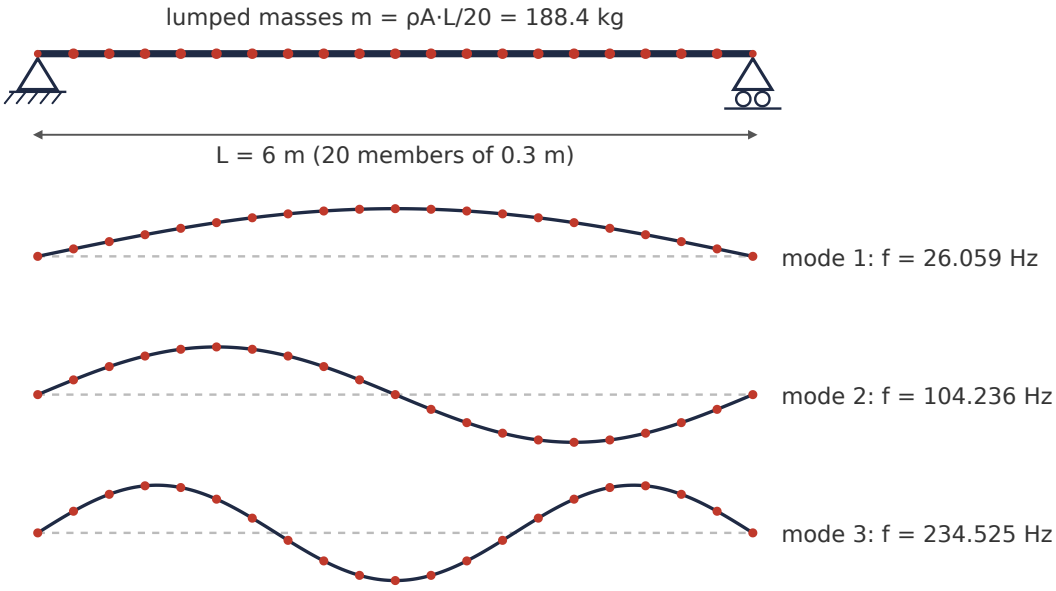

Fig. 1 — Top: the beam, its supports and the 21 lumped masses. Bottom: the first three vertical mode shapes — the exact sine (line) and the FEM mode shape at the nodes (dots).

1. Structure and mass

A single steel beam, L = 6 m, rectangular section R 200×400 (b = 200 mm, h = 400 mm, S235, E = 210 GPa, ρ = 7 850 kg/m³), bending about its strong axis in the vertical plane. A = 0.08 m², $I_y = bh^3/12 = 1.066667\cdot10^{-3}$ m⁴, $I_z = hb^3/12 = 2.666667\cdot10^{-4}$ m⁴. Node 1 is a pin (u_x, u_y, u_z and torsion fixed), node 21 a

roller (u_y, u_z and torsion fixed, free to slide along the axis). The same support set as the published UDL-beam example, so the beam is simply supported for bending in both planes.

The vibrating mass is the beam's own: load case 1 is self weight, and the DYN combination "DYN mass = self weight" (factor 1.0) is the mass source of the modal analysis. The calculator lumps it at the nodes as m = $|F_z|/g$ — each node receives the weight of the half-members on either side of it. Rotary inertia is omitted, as it is in the Euler-Bernoulli formula.

2. Hand solution

Data: L = 6 m, E = $210\cdot10^6$ kN/m², ρ = 7 850 kg/m³, A = 0.08 m², N = 20 members of h = 0.3 m. Units: kN, m, tonnes (1 t = 1 kN·s²/m), so ω² comes out in rad²/s².

> $EI_y = 210\cdot10^6$ kN/m² · $1.066667\cdot10^{-3}$ m⁴ = **224 000 kN·m²**; $EI_z = 210\cdot10^6 \cdot 2.666667\cdot10^{-4}$ = **56 000 kN·m²**
>
> $\bar{m} = \rho\cdot A$ = 7 850 kg/m³ · 0.08 m² = **628 kg/m** = 0.628 t/m; total mass M = $\bar{m}\cdot L$ = **3768 kg**
>
> Lumped masses: m = $\bar{m}\cdot h$ = 628 kg/m · 0.3 m = **188.4 kg** at nodes 2–20, m/2 = **94.2 kg** at nodes 1 and 21 (1)

2.1 Continuous beam. The textbook solution of $\bar{m}\cdot\ddot{w} + EI\cdot w'''' = 0$ with $w = w'' = 0$ at both ends has the modes $w_n = \sin(n\pi x/L)$ and

> $\omega_n = (n\pi/L)^2\cdot\sqrt{(EI/\bar{m})} \;\rightarrow\; f_n = n^2\pi/(2L^2)\cdot\sqrt{(EI/\bar{m})}$ (2)
>
> $f_1 = \pi/(2\cdot6^2\ \text{m}^2)\cdot\sqrt{(224\,000\ \text{kN·m}^2 / 0.628\ \text{t/m})} = 0.0436332\ \text{m}^{-2}\cdot 597.2335\ \text{m}^2/\text{s} =$ **26.059229 Hz**
>
> $f_2 = 4\cdot f_1 =$ **104.236916 Hz**; $f_3 = 9\cdot f_1 =$ **234.533062 Hz**; $T_1 = 1/f_1$ = 38.3741 ms

2.2 The lumped-mass model, exactly. The FEM does not solve the continuous beam but its discretisation: exact beam stiffness (the cubic element reproduces Euler-Bernoulli bending exactly between nodes) with the mass in 19 points. That model has a closed-form solution too. Its flexibility at the nodes is the beam's Green's function, which expands in the same sines, $G(x_i, x_j) = \Sigma_k\ 2/(L\cdot EI\cdot(k\pi/L)^4)\cdot\sin(k\pi x_i/L)\cdot\sin(k\pi x_j/L)$. Sampled at N equal spaces the sines are discretely orthogonal except where they alias — $\sin(k\pi j/N) = \pm\sin(n\pi j/N)$ for $k = 2Np \pm n$ — so the nodal sine vector $\sin(n\pi x_j/L)$ is an exact mode of the lumped model, with

$\omega_n^2 = \pi^4 \cdot EI/(\bar{m} \cdot L^4) \,/\, S_n, \quad S_n = \Sigma_{p\geq 0}\,[\,(2Np+n)^{-4} + (2N(p+1)-n)^{-4}\,]$ (3)

$N = 20$: $S_1 = 1 + 39^{-4} + 41^{-4} + 79^{-4} + \ldots = 1.000000850543 \;\rightarrow\; f_1 =$ **26.059217991 Hz**

$S_2 = 2^{-4} + 38^{-4} + 42^{-4} + \ldots = 0.062500865608 \;\rightarrow\; f_2 =$ **104.236194475 Hz**

$S_3 = 3^{-4} + 37^{-4} + 43^{-4} + \ldots = 0.012346570165 \;\rightarrow\; f_3 =$ **234.524597424 Hz**

The first term of S_n alone is the continuous beam (2); the aliased terms are the whole discretisation error. They add flexibility, so the lumped model always vibrates slightly *below* the continuous beam, and they fall as $(n/2N)^4$ — the h^4 convergence of section 6.

2.3 Weak axis. With mass in the horizontal direction too, the beam also bends about z. Same mass, same supports, $I_z = I_y/4$:

$f_{y,n} = f_{z,n} \cdot \sqrt{(I_z/I_y)} = f_{z,n}/2$: $f_{y,1} =$ **13.029608996 Hz**, $f_{y,2} = 52.118097$, $f_{y,3} = 117.262299$, $f_{y,4} = 208.449077$ Hz (4)

2.4 Axial mode. Along the axis the beam is a rod fixed at the pin and free at the roller. Continuous: $f = \sqrt{(E/\rho)}/(4L)$. The lumped chain of 20 springs EA/h, with m at nodes 2–20 and m/2 at the free end, is half of a fixed–fixed chain of 40 springs, whose modes are known exactly:

$\sqrt{(E/\rho)} = \sqrt{(210 \cdot 10^9\ \text{Pa} / 7\,850\ \text{kg/m}^3)} = 5172.194$ m/s; continuous $f_{ax} = 5172.194 / (4 \cdot 6) =$ **215.508090 Hz** (5)

lumped: $f_{ax} = N/(\pi L) \cdot \sqrt{(E/\rho)} \cdot \sin(\pi/(4N)) = 20/(\pi \cdot 6) \cdot 5172.194 \cdot \sin(\pi/80) =$ **215.452703887 Hz** (6)

2.5 Effective modal masses. For a mode φ the effective mass in direction d is $M_{eff} = (\Sigma\, m_j \varphi_j)^2 / \Sigma\, m_j \varphi_j^2$. With $\varphi_j = \sin(n\pi j/N)$ the sums are $\Sigma \sin = \cot(n\pi/2N)$ for odd n (zero for even n) and $\Sigma \sin^2 = N/2$:

bending, odd n: $M_{eff,n} = (2m/N) \cdot \cot^2(n\pi/2N)$: $M_{eff,1} = (2 \cdot 188.4/20) \cdot \cot^2(\pi/40) =$ **3041.673515 kg**; $M_{eff,3} =$ **326.868766 kg** (7)

axial: $M_{eff} = (m/2N) \cdot \cot^2(\pi/4N) = (188.4/40) \cdot \cot^2(\pi/80) =$ **3051.086244 kg** (8)

continuous beam: $M_{eff,1} = 8/\pi^2 \cdot M = 0.810569 \cdot 3\,768\ \text{kg} = 3054.226$ kg

The lumped value is 0.4 % below the continuous one mainly because the 2 × 94.2 kg at the supports sit on restrained DOFs and never move.

3. FEM model

The structure is stored as a regular project file, `verification-beam-natural-frequencies.json`. The beam is split into 20 members of 0.3 m (21 nodes); load case 1 is self weight (factor 1.0), and the DYN combination of that case is the mass source. Shear deformation is off (Settings → Solver), because the reference is Euler-Bernoulli. The modal analysis (`modalAnalysis()`, Dynamics → Modal analysis in the app) was run twice — with the mass acting in Z only (the vertical modes of section 2.1–2.2) and in X, Y and Z (all modes) — each time with both eigensolvers, subspace iteration and shift-invert Lanczos. Values are the solver's own, unrounded.

4. Results comparison

4.1 Lumped masses

Quantity	Formula	Hand solution	FEM result	Rel. difference
Total mass M [kg]	ρ·A·L	3 768.000	3 768.000000000	$< 10^{-15}$
Mass moving vertically [kg]	19·m	3 579.600	3 579.600000000	$< 10^{-15}$
Mass moving axially [kg]	19·m + m/2	3 673.800	3 673.800000000	$< 10^{-15}$

4.2 Vertical modes (mass in Z)

Quantity	Formula	Hand solution	FEM result	Rel. difference
Against the lumped-mass closed form — subspace iteration				
f_1 [Hz]	eq. (3)	26.059217991	26.059217991268	7×10^{-13}
f_2 [Hz]	eq. (3)	104.236194475	104.236194475448	6×10^{-14}
f_3 [Hz]	eq. (3)	234.524597424	234.524597424008	6×10^{-13}
Against the lumped-mass closed form — Lanczos				
f_1 [Hz]	eq. (3)	26.059217991	26.059217991268	7×10^{-13}
f_2 [Hz]	eq. (3)	104.236194475	104.236194475448	6×10^{-14}
f_3 [Hz]	eq. (3)	234.524597424	234.524597423839	2×10^{-13}
Against the continuous beam (discretisation error of 20 members)				
f_1 [Hz]	eq. (2)	26.059229	26.059217991	4×10^{-7}
f_2 [Hz]	eq. (2)	104.236916	104.236194475	7×10^{-6}

f_3 [Hz]	eq. (2)	234.533062	234.524597424	4×10^{-5}
Period T_1 [ms]	$1/f_1$	38.374121	38.374136950	4×10^{-7}

The frequencies are the lumped-mass model's exactly — the differences are round-off — and every one lies below the continuous beam, as section 2.2 predicts. The two eigensolvers, which share nothing but the factorised stiffness matrix, agree to 10^{-12}.

✓ Vertical frequencies match the exact solution of the model to round-off VERIFIED

4.3 All modes (mass in X, Y and Z), in the order the solver returns them

Mode	Shape	Dir.	Hand solution [Hz]	FEM result [Hz]	Rel. difference
1	lateral (weak axis), n = 1	Y	13.029609	13.029608996	7×10^{-13}
2	vertical, n = 1	Z	26.059218	26.059217991	7×10^{-13}
3	lateral, n = 2	Y	52.118097	52.118097238	6×10^{-14}
4	vertical, n = 2	Z	104.236194	104.236194475	6×10^{-14}
5	lateral, n = 3	Y	117.262299	117.262298712	2×10^{-13}
6	lateral, n = 4	Y	208.449077	208.449077446	5×10^{-13}
7	axial (fixed–free rod)	X	215.452704	215.452703887	2×10^{-13}
8	vertical, n = 3	Z	234.524597	234.524597424	2×10^{-12}

Hand values from eq. (3) with I_y or I_z, and eq. (6) for the axial mode. Every weak-axis frequency is exactly half the strong-axis one of the same n (eq. 4), and the axial mode, at 215.453 Hz, slots in between the fourth lateral and the third vertical bending mode — the solver finds all eight, in order, with no mode skipped (its Sturm-sequence check confirms the count). Torsion has no mode: rotary inertia is not part of the mass.

4.4 Effective modal masses

Quantity	Formula	Hand solution	FEM result	Rel. difference
$M_{eff,1}$ vertical [kg] — subspace	eq. (7)	3 041.673515	3 041.673514947	3×10^{-15}
$M_{eff,1}$ vertical [kg] — Lanczos	eq. (7)	3 041.673515	3 041.673514947	4×10^{-15}
$M_{eff,2}$ vertical [kg]	0 (antisym.)	0.000000	0.000000000	0
$M_{eff,3}$ vertical [kg] — Lanczos	eq. (7)	326.868766	326.868765772	2×10^{-14}
M_{eff} axial [kg] — Lanczos	eq. (8)	3 051.086244	3 051.086244037	8×10^{-12}
$M_{eff,1}/M$ vs continuous beam	$8/\pi^2$	0.810569	0.807238194	4×10^{-3}

Lanczos (with full reorthogonalisation) reproduces the effective masses to round-off. Subspace iteration stops once every eigenvalue changes by less than 10^{-7} between sweeps; its frequencies are then exact, but its mode shapes — and hence effective

masses — carry an error of order 10^{-8}, far below anything that matters for a response-spectrum analysis.

✓ Mode order, weak-axis and axial modes and effective masses match the closed forms VERIFIED

5. Final results

Final value	Formula	Hand solution	FEM result	Difference
First natural frequency f_1 [Hz]	$\pi/(2L^2)\cdot\sqrt{(EI/\bar{m})}$	26.059229	26.059217991	4×10^{-7}
First period T_1 [ms]	$1/f_1$	38.374121	38.374136950	4×10^{-7}
Second frequency f_2 [Hz]	$4\cdot f_1$	104.236916	104.236194475	7×10^{-6}
Third frequency f_3 [Hz]	$9\cdot f_1$	234.533062	234.524597424	4×10^{-5}
Effective mass, mode 1 [%]	$8/\pi^2$	81.0569	80.723819	4×10^{-3}

The effective-mass row compares the lumped model with the continuous beam: the 0.4 % is the mass that sits on the two supports (section 2.5), not a solver error — against eq. (7) it agrees to 10^{-15}.

✓ Final: f_1 = 26.059 Hz (T_1 = 38.37 ms), f_2 = 104.236 Hz, f_3 = 234.525 Hz — hand and FEM agree VERIFIED

6. Mesh convergence

The same beam solved with 2, 4, 10, 20 and 40 members (vertical mass only), against the continuous beam (2):

Members	f_1 FEM	rel. error	f_2 FEM	rel. error	f_3 FEM	rel. error
2	25.870060	7×10^{-3}	—		—	
4	26.051279	3×10^{-4}	103.480240	7×10^{-3}	219.710991	6×10^{-2}
10	26.059049	7×10^{-6}	104.224539	1×10^{-4}	234.375305	7×10^{-4}
20	26.059218	4×10^{-7}	104.236194	7×10^{-6}	234.524597	4×10^{-5}
40	26.059228	3×10^{-8}	104.236872	4×10^{-7}	234.532553	2×10^{-6}
exact	**26.059229**		**104.236916**		**234.533062**	

With 2 members there is a single moving mass and so a single vertical mode. Every doubling of the mesh cuts the error by about 16×, the h^4 rate eq. (3) predicts; higher modes need more members, since mode n has n half-waves to describe — mode 3 is within 1 % from 10 members. All values are below the continuous beam.

7. Conclusion

The FEM calculator's modal analysis reproduces the exact solution of its lumped-mass model to floating-point round-off — frequencies and mode order across three directions with both eigensolvers, effective masses with Lanczos (to 10^{-8} with subspace iteration, whose convergence test is on the eigenvalues). Against the continuous Euler-Bernoulli beam the first frequency is within $4 \cdot 10^{-7}$ on 20 members (26.059218 Hz against 26.059229 Hz), and the error falls with the fourth power of the element length, always from below.

The same comparison runs as part of the automated test suite on every test run: `tests/verificationExamples.test.ts` solves the project file `calc_tests/verification/beam-natural-frequencies.json` and asserts its lumped masses, frequencies, mode order and effective masses against the closed forms above, recomputed from the section and material stored in the file.